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TO WHOM IT MAY CONCERN

Animikh Roy is known to me for last three years during which he has been engaged in different internship programme with me over a period of nearly one year.

His area of work with me was based on ‘Aspects of General Theory of Relativity and black holes'. During this study Animikh has covered the following topics: Tensors in curved space time, Riemannian geometry, Einstein's equation, Tests of Einstein's theory, black hole solutions, Kruskal coordinates, black holes, Penrose diagramme. In this short time Animikh has learned most of the essential ideas in this area very well which is indeed very commendable.

I was extremely impressed with his performance as he did all the calculations independently and raised many critical questions which reflected his strong analytic mind. He is a very intelligent and hard working student and through my long interactions with him on various science topics, I can recognize that Animikh has a strong potential to become a valuable researcher in the field of Theoretical Physics and interdisciplinary sciences.

Animikh is very interactive and has the ability to communicate his thoughts very clearly. He can present his works with confidence and can plan future directions. In fact currently he is engaged in a study on 'Consciousness' from both Physics and biological approaches at the National Institute for Advanced studies (NIAS) in Bangalore, India. For this, he not only applies his Physics knowledge in quantum mechanics but also undertakes some further studies in neurobiology at the National Centre for Biological Sciences (NCBS) at Bangalore. I believe that such complex issues can only be resolved using the methods of such different and complimentary areas of science.

This interdisciplinary approach is his valuable quality and I strongly hope that he will turn out to be a very competent scientist in coming days.

In addition to these Animikh has a nice personality who can work in a group with a positive mind. Moreover, I found that he is never afraid of taking new challenges.

At this stage Animikh needs to be in a vibrant and inspiring Graduate programme in one of the well-known universities and I am sure that he will be a valuable addition as a student.

I therefore have no hesitation in recommending him very strongly for Graduate studies in the department of Physics of your University.

Yours sincerely,

Soumitra SenGupta

Soumitra SenGupta

PHYSICS
OF
BLACK HOLES

By

Animikh Roy

Indian Association for the Cultivation of Science

Advisor: Prof. Soumitra Sengupta

ABSTRACT

This project mostly focuses on a theoretical development of the aspects of Black Hole mechanics and thermodynamics. It involves an overall glimpse on the preliminary ideas including the fundamental Equivalence Principle, the description of Flat to Curved Space-time, the physical basis of the Geodesic equation, followed by brief ideas about the Riemann Curvature Tensor, Einstein Field Equations, the Schwarzschild metric, Kruskal-Szekeres coordinates and Penrose diagrams. However, the prime focus of this project include the study and theoretical analysis of charged Black Holes with the help of Reissner-Nordström coordinates and rotating Black Holes involving the Kerr solution and the Penrose Process. The Final portion of this project deals with the Penrose diagrams of each individual solution along with the Penrose Process of Energy Extraction, which paves a path for a thermodynamic approach of studying Black Hole phenomenology.

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7) Charged Black Holes:

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INTRODUCTION

General relativity or the general theory of relativity is a geometric view on the Theory of gravitation published by Albert Einstein in 1916. It is currently regarded as the most accurate description of gravitation in modern physics. General relativity generalises special relativity and Newton's law of universal gravitation and provides a unified description of gravity as a geometric property of space and time. In particular, the curvature of space-time is directly related to the energy and momentum of whatever matter and radiation are present. The predictions of general relativity have been confirmed in all observations and experiments to date. General Relativity has been accurately tested in the solar system. It underlies our understanding of the universe on the largest scales and is central to the explanation of such frontier astrophysical phenomena as gravitational collapse, black holes and the big bang theory. Although General relativity is not the only relativistic theory of gravity, it is the simplest and the most elegant geometric theory that is consistent with experimental data.

This project will mainly focus on the idea that Gravity is the geometry of four-dimensional curved Space-time; including the elegant description of curved Space time in relation to Local inertial frames and Geodesics. In order to lay the foundation to study Black Holes in detail we shall explore the concept of the curvature tensor and derive Einstein's field equations from the Principle of Action. Essentially we shall deal with the development of ideas about the charged and rotating Black Holes along with their various characteristic features; starting from the Schwarzschild solution for spherically symmetric gravitational collapse, we shall investigate up to the Kerr solution along with all their respective Penrose diagrams. In the final section we shall also deal with the Penrose process of Energy extraction from rotating Black Holes which leads to a thermodynamic approach to study this unique compact object.

In a nutshell before going into details, we know that Black holes are the most fascinating objects predicted by general relativity. A black hole is an object so dense that it sufficiently bends the space-time around it so that nothing can escape. In other words, if an object is so dense that its escape velocity is greater than the speed of light, then nothing can escape the gravity of that object and it is called a black hole. The maximum distance from the center of the black hole for which nothing can escape is called the event horizon. There exist about 20 confirmed candidates for astrophysical Black Holes in the mass range 5 - 20 $M_\odot$ and about three dozen super massive Black Hole candidates in the mass range 106 - 109.5 $M_\odot$. Unfortunately, there exists as yet no direct evidence for astrophysical Black Holes. At present we only hope that the black-hole paradigm may be proved or ruled out by comparing Black Hole candidates with credible alternatives. Fortunately, Black Holes are dark and compact, which narrows the list of possible alternatives among standard astrophysical objects. So let us now consider the origin of this wonderful concept leading to a richer and more profound understanding of our Universe through the study of Gravity and its physical and Mathematical build-up in the hands of Einstein, Riemann, Schwarzschild and others.

Gravity introduces General Relativity in a different order. The simplest physically relevant solutions are presented first followed by the observational consequences which have been explored through the study of the motion of test particles and light rays in them. However, this project mainly stresses on the physical and mathematical background leading to the mechanics and thermodynamics of Black Holes.

Chapter 1

GRAVITY AS GEOMETRY

Gravitational Mass and Inertial Mass

In dealing with the notion of Gravity, let us first consider the aspects of Gravitational mass and Inertial Mass. A subtlety arises when we compare the law of universal gravitation with Newton's second law of motion. The mass that appears in the law of universal gravitation is the property of the particle that creates the gravitational force acting on the other particle; for if we double $\boldsymbol{m_2}$, we double the force on $\boldsymbol{m_1}$. Similarly, the mass in the law of universal gravitation is the property of the particle that responds to the gravitational force created by the other particle. The law of universal gravitation provides a definition of gravitational mass as the property of matter that creates and responds to gravitational forces. Newton's second law of motion, **F=ma**, describes how any force, gravitational or not, changes the motion of an object. For a given force, a large mass responds with a small acceleration and vice versa. The second law provides a definition of inertial mass as the property of matter that resists changes in motion or, equivalently, as an object's inertia. Is the inertial mass of an object necessarily the same as its gravitational mass? This question troubled Newton and many others since his time. Experiments are consistent with the premise that inertial and gravitational mass are the same. We can measure the weight of an object by suspending it from a spring balance. Earth's gravity pulls the object down with a force (weight) of $\boldsymbol{m_g\, g}$, where g is the local gravitational acceleration and $\boldsymbol{m_g}$ the gravitational mass of the object.

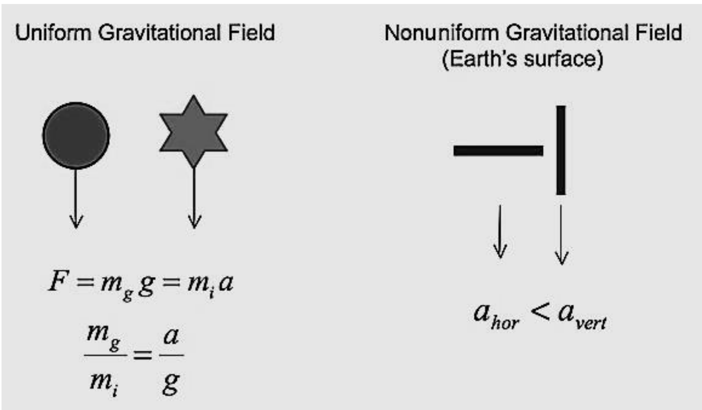


Fig.1(a)Compares uniform and non uniform Gravitational fields

Gravity's pull on the object is balanced by the upward force provided by the stretched spring. We say that two masses that stretch identical springs by identical amounts have the same gravitational mass, even if they possess different sizes, shapes, or compositions. But will they have the same inertial mass? We can answer this question by cutting the springs, letting the masses fall, and measuring the accelerations. The second law says the net force acting on the mass is the product of the inertial mass, $\boldsymbol{m_i}$, and acceleration, **a**, giving us $\boldsymbol{m_g g} = \boldsymbol{m_i a}$ or $\frac{g}{a} = \boldsymbol{m_i/m_g}$ But g is a property of the Earth alone and does not depend upon which object is placed at its surface, while experiments find the acceleration, a, to be the same for all

objects falling from the same point in the absence of air friction. Therefore**, g/a** is the same for all objects and thus for $\boldsymbol{m_{i/}m_g}$. We define the universal gravitational constant, G, to make$\boldsymbol{m_i = m_g}$.
The principle of the universality of free fall is the statement that all materials fall at the same rate in a uniform gravitational field. This principle is equivalent to the statement that $\boldsymbol{m_i = m_g}$. Physicists have found the principle to be valid within the limits of their experiments' precision, allowing them to use the same mass in both the law of universal gravitation and Newton's second law.

Principle of Equivalence

There are several ways to formulate the Principle of Equivalence, but one of the simplest is Einstein's original insight: he suddenly realized, while sitting in his office in Bern, Switzerland, in 1907, that if he were to fall freely in a gravitational field, he would be unable to feel his own weight. Einstein later recounted that this realization was the "happiest moment in his life", for he understood that this idea was the key to how to extend the Special Theory of Relativity to include the effect of gravitation. A little reflection will show that the law of the equality of the inertial and gravitational mass is equivalent to the assertion that the acceleration imparted to a body by a gravitational field is independent of the nature of the body. For Newton's equation of motion in a gravitational field, written out in full, is:

$$\textbf{(Inertial mass) . (Acceleration)} = \textbf{(Intensity of the gravitational field) . (Gravitational mass).}$$

It is only when there is numerical equality between the inertial and gravitational mass that the acceleration is independent of the nature of the body. This constitutes the central essence of the Principle of Equivalence which remains as one of the founding pillars of General theory of Relativity.

The greatest importance of the Principle of Equivalence lies in the fact that at any local region in space-time it is possible to formulate the laws governing various physical principles neglecting the effect of gravity. This clearly indicates that special theory of Relativity which does not involve gravity is valid in such regions of space-time. The above explanation also clearly shows us that that inertial fields and gravitational fields are equivalent and interchangeable from the point of view of the selected inertial frame and the observer.

Clocks in a Gravitational Field

When comparing a clock under the influence of gravitational forces with one clock very far from such influences it is found that the first clock is slow compared to the second. To see this consider the same clock we used in the Special Theory of Relativity. For this experiment, however, imagine that the clock is being accelerated upward, being pulled by a crane. The clock gives off a short light pulse which moves towards the mirror at the top of the box, at the same time the mirror recedes from the pulse with even increasing speed (since the box accelerates). Still the pulse eventually gets to the mirror where it is reflected, now it travels downward where the floor of the box is moving up also with ever increasing velocity. On the trip up the distance covered by light is larger than the height of the box at rest, on the trip down the distance is smaller. A calculation shows that the whole distance covered in the trip by the pulse is larger than twice the height of the box, which is the distance covered by a light pulse when the clock is at rest. Thus we can conclude that time slows down under the influence of intense gravitational fields as light bends under the influence of Gravity. Fig 1 (b) Shows a thought experimental diagram to describe the gravitational slowing down of time as described below.

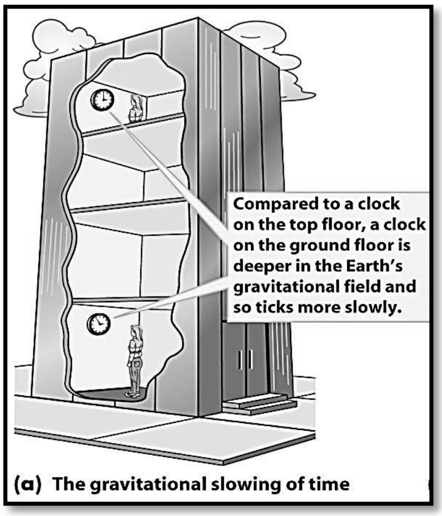


Fig.1 (b)

Minkowski Space and Light Cones

Einstein's physical intuition motivated his formulation of special relativity, but his generalization to general relativity would not have occurred without the mathematical formulation given by Hermann Minkowski. In 1907, Minkowski realized the physical notions of Einstein's special relativity could be expressed in terms of events occurring in a universe described with a non-Euclidean geometry. Minkowski took the three spatial dimensions with an absolute time and transformed them into a 4dimensional representation called space-time.

The nature of Space-time is considered flat in Special Theory of Relativity and we shall investigate how space-time curves under the influence of matter and gravity in the later sections of the project. As of now in flat Minkowski Space-time we require 4 coordinates to describe an event. These coordinates are usually taken to be a time coordinate and three spatial coordinates. While we could denote these as **(t, x, y, z)** instead we use index notations such as the following to denote the respective coordinates more frequently $(\boldsymbol{x^0},\boldsymbol{x^1},\boldsymbol{x^2},\boldsymbol{x^3})$ where x^0 refers to the time coordinate. Here **x** is a positive 4-vector having the dimensions of length. Additionally so that we have the same units for all coordinates, we measure time in terms of space, by multiplying time with the speed of light such that $\boldsymbol{x = c.t}$. This Space-Time which Minkowski formulated is known as Minkowski-Space where we define an invariant interval between two events a and b in space-time as:

$$\boldsymbol{S^2 = -(x_a^0 - x_b^0)^2 + (x_a^1 - x_b^1)^2 + (x_a^2 - x_b^2)^2 + (x_a^3 - x_b^3)^2}$$

So, $\boldsymbol{ds^2} = -(\boldsymbol{dx^0})^2 + (\boldsymbol{dx^1})^2 + (\boldsymbol{dx^2})^2 + (\boldsymbol{dx^3})^2$ (1.1)

It is invariant because another observer using the coordinate system $(\boldsymbol{x'^0}, \boldsymbol{x'^1}, \boldsymbol{x'^2}, \boldsymbol{x'^3})$ would measure the same interval, that is

$$\boldsymbol{ds'^2} = -\left(\boldsymbol{dx'^0}\right)^2 + \left(\boldsymbol{dx'^1}\right)^2 + \left(\boldsymbol{dx'^2}\right)^2 + \left(\boldsymbol{dx'^3}\right)^2$$

This invariant interval is analogous to distance in flat 3 dimensional space that we are accustomed to. However this distance as we see can be negative. We separate the intervals into three types:

$\boldsymbol{ds^2} > 0$ The interval is **space-like**
$\boldsymbol{ds^2} < 0$ The interval is **time-like**
$\boldsymbol{ds^2} = \boldsymbol{0}$ The interval is **light-like**

A space-like interval is one for which an inertial frame can be found such that two events are simultaneous. No material object can be present at two events which are separated by a space-like interval. However, a time-like interval can describe two events of the same material object. If a ray of light could travel between two events then we say that the interval is light-like.

The Space-time diagram which corresponds to the description of such various events geometrically appear in the shape of a cone and is called a "Light-cone" as is represented below:

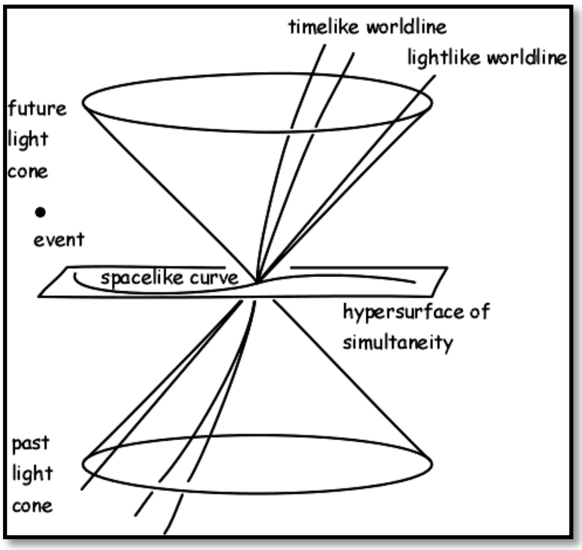


Fig.1 (c)

In the Fig.1(c) representing a Minkowski space-time diagram showing the past and future light cones we have considered 2 dimensions of Space and 1 dimension of time. As illustrated in the diagram, for a single object, we define the set of all past and future events of that object as the **worldline** of that object. Thus, if two events are on the worldline of a material object, then they are separated by a time-like interval. If two events are on the worldline of a photon, then they are separated by a light-like interval.

The "hyper-surface" mentioned in the above diagram represent spatial snapshots of space-time. Now in this figure the hyper-surface is 2 dimensional but in reality we consider 3 dimensional hyper surfaces for 4 dimensional space-time. The in the hyper-surface of simultaneity all events occur with zero time like separation. Therefore all events only have space like separation and they are causally disconnected. This means that considering the fact that nothing can travel faster than the speed of light, whatever occurs in the region beyond the light cone occurs in the present. If they are on the same hyper-surface then they are simultaneous, but even if they are not, every event is independent of each other and one's present state does not affect the future of another event.

Now using Einstein's summation conventions we can write the above equation (1.1) as

$$\boldsymbol{ds^2} = \eta_{\mu\boldsymbol{\nu}}\boldsymbol{dx}^{\mu}\boldsymbol{dy}^{\boldsymbol{\nu}} \qquad (1.2)$$

for $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ values of ***{0, 1, 2, 3}*** where we implement Einstein's summation notation. This notation is a simple way in which to condense many terms of a summation. For instance, the above equation could be written as 16 terms:

$$\boldsymbol{ds^2} = \eta_{00}\boldsymbol{dx^0dx^0} + \eta_{01}\boldsymbol{dx^0dx^1} + \eta_{02}\boldsymbol{dx^0dx^2} + \eta_{03}\boldsymbol{dx^0dx^3} + \eta_{10}\boldsymbol{dx^1dx^0} + \ldots \qquad (1.3)$$

or more simply as

$$\boldsymbol{ds^2} = \sum_{\boldsymbol{0}}^{\boldsymbol{3}}\sum_{\boldsymbol{0}}^{\boldsymbol{3}} \eta_{\mu\boldsymbol{\nu}}\,\boldsymbol{dx}^{\mu}\boldsymbol{dy}^{\boldsymbol{\nu}}$$

In Einstein's summation notation we simply note that when a variable is repeated in the upper and lower index of a term, then it represents a summation over all possible values. In the above case, μ and ν are in the lower indices of η and the upper indices of x and so we know to sum over all possible values of μ and ν, which in this case would give us 16 terms.

We mentioned that this is another expression for equation (1.1), but we see that equation (1.1) only has four nonzero terms. Then we must constrain the values of $\eta_{\mu\nu}$ such that:

$$\eta_{\mu\nu} = \begin{pmatrix} \eta_{00} & \eta_{01} & \eta_{02} & \eta_{03} \\ \eta_{10} & \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{20} & \eta_{21} & \eta_{22} & \eta_{23} \\ \eta_{30} & \eta_{31} & \eta_{32} & \eta_{33} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (1.4)$$

The matrix $\eta_{\mu\nu}$ is referred to as the **metric tensor** for Minkowski space.

Curved Space-Time under Gravity

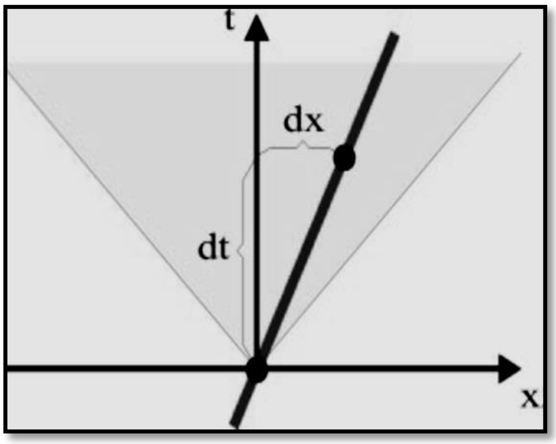


Figure 1(d)

The deflection of light in a gravitational field suggests that gravity is associated with the curvature of space-time. To begin with the concept of curved space-time and its illustrations let us first consider two events on a moving clock, separated by a time **dt** and a distance **dx**, as seen relative to the system at rest. We can illustrate the two events, and the motion of the clock in a space-time diagram as depicted in Fig. 1(d).

Time is directed upwards in the diagram. The motion of the clock corresponds to a world-line in the diagram. The proper time interval **dτ** is the time betwee ; according to the moving clock, which is given by:

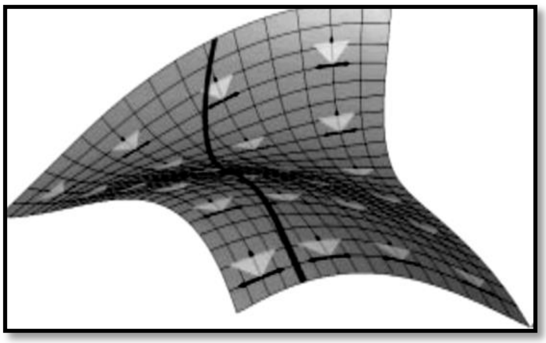
$$d\tau^2 = dt^2 - \left(\frac{dx}{c}\right)^2$$

Here c is the velocity of light. In the above limit as the speed of the clock approaches the speed of light we have $dx = cdt$, and thus $d\tau = 0$.

A clock moving almost at the speed of light will thus almost not tick at all relative to the clocks at rest. It is customary to choose the axes of the space-time diagram in such a manner that motion at the speed of light corresponds to a line that is inclined at a 45° angle relative to the axes of the diagram. At every point in the diagram we can then draw a little triangle, with a 90° opening angle, known as a light-cone. No material objects can travel faster than the velocity of light, which means that the world-lines of objects must always be directed within the local light-cone.

In general relativity we have a curved space-time, which we may illustrate by a curved surface with little locally flat coordinate systems, known as Minkowski systems, living on it as illustrated in Fig.1(e)

Figure 1 (e)



The little coordinate systems on the surface work precisely as the space-time diagram of Fig. 1(e). In particular the worldliness of moving objects must always be directed within the local light cone. To find out how much a clock has ticked along its winding world-line, we consider nearby events along the world-line and sum up the **d**τ's from the above calculated relations for the local Minkowski Systems under consideration.

Chapter 3

CURVED SPACE-TIME

Arriving At the Metric

So far we have shown that in flat Minkowski Space-time the distance between two events is given by:

$$ds^2 = \eta_{\mu\nu} dx^{\mu} dy^{\nu} = dt^2 - dx^2 \tag{2.1}$$

Also this interval between 2 events can be represented in terms of proper time as:

$$d\tau^2 = \eta_{\mu\nu} dx^{\mu} dy^{\nu} \tag{2.2}$$

Where $\eta_{\mu\nu}$ is the matrix defined in the relation (1.4). However the essence of General Theory is that Gravity creates curvature so to include gravity, we need to modify our assumption of flatness (Minkowski space) and work in a more general *curved* space. Here the interval between two events is given by:

$$ds^2 = g_{\mu\nu} dx^{\mu} dy^{\nu} \tag{2.3}$$

Where $g_{\mu\nu}$ is a symmetric 4×4, position dependent matrix (the metric). It has to be non-singular so that g^{-1} exists everywhere. The form of $g_{\mu\nu}$ will be different in different coordinate systems for the same geometry. Since there are 4 arbitrary functions involved in transforming 4 coordinates, there are only 10-4 = 6 independent functions associated with a metric.

The Summation Convention and Tensors

So far we have used a certain method in index notations that we shall follow throughout this article. The outlining rules of this method are as follows:

1. The location of the indices must be respected: superscripts (upper indices) for coordinates and vector components and subscripts (lower indices) for the metric. (In expressions such as the chain rule, $dx^{\alpha} = \left(\frac{\partial x^{\alpha}}{\partial x'^{\beta}}\right) dx'^{\beta}$, the superscript β in the denominator acts as a subscript.)
2. Repeated indices always occur in superscript-subscript pairs and imply summation. For that reason they are called summation indices. One index is as good as any other for indicating a summation, and for this reason summation indices are also called dummy indices. Thus, $g_{\alpha\beta} a^{\alpha} b^{\beta}$means the same thing as $g_{\gamma\delta} a^{\gamma} b^{\delta}$. Expressions with three or more repeated indices, such as $g_{\alpha\alpha} a^{\alpha} b^{\alpha}$, or repeated indices that are not in superscript-subscript pairs, such as $g_{\alpha\beta} g_{\beta\gamma}$, , will never occur.
3. Indices that are not summed are called free indices. They must balance on both sides of an equation. The value of a free index can be changed if it is changed on both sides of an equation at the same time. The equation:

$$g_{\alpha\beta} = g_{\beta\alpha} \tag{2.4}$$

expresses the symmetry of the metric. The indices balance because there is one lower index, α and β, on each side of the equation. An equation such as this can be thought of as a shorthand for an array of equations for each of the four possible values of the free indices α and β

Equation (2.4) stands for the 16 equations:

$$g_{00} = g_{00},\ g_{01} = g_{10}\ ,\ g_{02} = g_{20}, \dots \qquad (2.5)$$
$$g_{10} = g_{01}, g_{11} = g_{11},\ g_{12} = g_{21}, \dots$$

For this reason, a free index can be changed to another free index (not already tied up in a summation) provided it is changed on both sides of an equation at the same time. Changing β to γ in (2.3) gives $g_{\alpha\gamma} = g_{\gamma\alpha}$ which represents the same set of 16 relations (2.5). An expression such as $g_{\alpha\beta} = g_{\alpha\gamma}$, in which the indices don't balance, is meaningless and cannot be used.

Light Cones and World Lines

We have already gone through the concept of Light Cones in the previous section involving the space time diagrams of Special Theory of Relativity. The world line of an object is the unique path of that object as it travels through 4-dimensional space-time. The concept of "world line" is distinguished from the concept of "orbit" or "trajectory" by the presence of its time dimension. It typically encompasses a large area of space-time wherein perceptually straight paths are calculated to show their (relatively) more absolute position states to reveal the nature of special relativity or gravitational interactions.

In other words, the worldline of an object is the sequence of space-time events corresponding to the history of the object. It is a time-like curve in space time where each of its points is an event that can be labelled with the time and the spatial position of the object at that time.

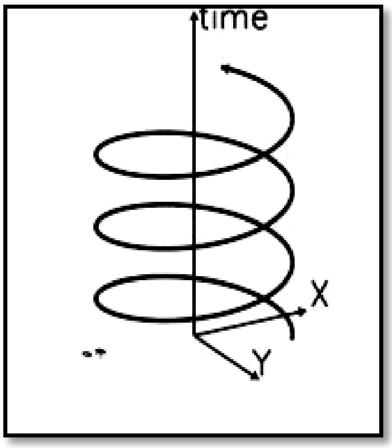


Fig.2(a)

For example, the orbit of the Earth in space is approximately a circle, a three-dimensional (closed) curve in space: the Earth returns every year to the same point in space. However, it arrives there at a different (later) time. The world line of the Earth is helical in space-time (a curve in a four-dimensional space) and does not return to the same point. This is example is represented in Fig.2 (a).

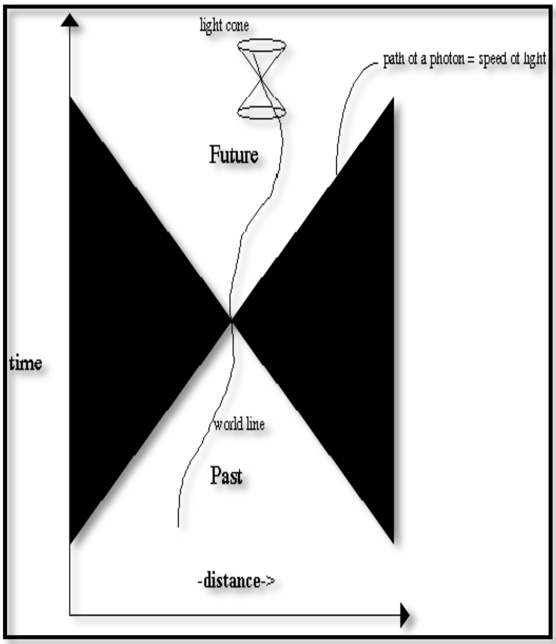


Fig.2(b)

The use of world lines in general relativity is basically similar to that in special relativity, with the difference that in this case, the structure of space-time can be curved. A metric exists and its dynamics are determined by the Einstein field equations which (will be described later in this article) and are dependent on the mass distribution in space-time. Again the metric defines light-like (null), space-like and time-like curves. Also, in general relativity, world lines are time-like curves in space-time, where time-like curves fall within the light cone. Here, a light cone is not necessarily inclined at 45 degrees to the time axis due to its curved path under the influence of gravity as shown in Fig. 2(b). However, any time-like curve admits a co-moving observer whose "time axis" corresponds to that curve, and, since no observer is privileged, we can always find a local coordinate system in which light-cones are inclined at 45 degrees to the time axis.

Vectors in Curved Space Time

Our key to defining vectors in curved space-time is to recognize that vectorial quantities-momentum, velocity, current density, etc. that is those which are represented by vectors-are all local. They can only be measured by an observer in a laboratory located in a small region of space-time. The way to define vectors in curved space-time is, therefore, to separate the notions of magnitude and direction and to define direction locally by means of small vectors, exactly as a physicist working in a local laboratory would. Vectors are thus defined at a point and there they obey all the usual flat space- time rules of vector algebra. An assignment of a vector to each point in space-time in a smooth way, a = a(x), is called a vector field. Vectors defined at different points, however, are in different tangent spaces, and there is no way of adding vectors at different points, as there is in flat space-time. Position vector is another notion that must be abandoned because it is not a local idea. Similarly, displacement vectors must be abandoned, except for the displacement vector between infinitesimally separated points, which is a. local quantity.

Thus in curved Space-time, a combination of vectors $\boldsymbol{x}^{\boldsymbol{\alpha}}$ in the basis $\boldsymbol{e}_{\boldsymbol{\alpha}}(\boldsymbol{x})$ is expressed as the linear combination: $\boldsymbol{a}(\boldsymbol{x}) = \boldsymbol{a}^{\boldsymbol{\alpha}}(\boldsymbol{x})\, \boldsymbol{e}_{\boldsymbol{\alpha}}(\boldsymbol{x})$ (2.6)

The numbers $\boldsymbol{a}^{\boldsymbol{\alpha}}(\boldsymbol{x})$ are called the components of the vector a in the basis $\boldsymbol{e}_{\boldsymbol{\alpha}}(\boldsymbol{x})$.

The scalar product between any two vectors a and b at the same point can be computed in terms of the components if the scalar products of the basis vectors are known in the following way:

$$\begin{aligned}\boldsymbol{a}.\boldsymbol{b} &= \boldsymbol{a}^{\boldsymbol{\alpha}}\boldsymbol{e}_{\boldsymbol{\alpha}}.\boldsymbol{b}^{\boldsymbol{\beta}}\boldsymbol{e}_{\boldsymbol{\beta}} \\ &= (\boldsymbol{e}_{\boldsymbol{\alpha}}.\boldsymbol{e}_{\boldsymbol{\beta}})\boldsymbol{a}^{\boldsymbol{\alpha}}\boldsymbol{b}^{\boldsymbol{\beta}}\end{aligned} \tag{2.7}$$

We can pick a basis in which the scalar products are anything we like, but two types of bases are of particular importance: (1)Orthonormal Bases (2)Coordinate Bases

Orthonormal Bases

An orthonormal basis consists of four mutually orthogonal vectors of unit length $\boldsymbol{e}_{\hat{\alpha}}.\boldsymbol{\alpha}$ = 0, 1, 2, 3. Where, hat on the index is used to distinguish the orthonormal bases and components from other kinds. In space-time three of the orthogonal unit vectors may be space-like but one must be time-like. The requirements for an orthonormal basis are, therefore, conveniently summarized by

$$\boldsymbol{e}_{\hat{\alpha}}(\boldsymbol{x}).\boldsymbol{e}_{\hat{\beta}}(\boldsymbol{x}) = \eta_{\hat{\alpha}\hat{\beta}} \tag{2.8}$$

Where $\eta_{\hat{\alpha}\hat{\beta}} = \boldsymbol{diag}(-\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$; so in terms of orthonormal basis components we get the scalar product between vectors in the form:

$$\boldsymbol{a}.\boldsymbol{b} = \eta_{\hat{\alpha}\hat{\beta}}\boldsymbol{a}^{\hat{\alpha}}\boldsymbol{b}^{\hat{\beta}} \tag{2.9}$$

The momentum in orthonormal basis is expressed as: $\boldsymbol{P} = \boldsymbol{p}^{\hat{\alpha}}\boldsymbol{e}_{\hat{\alpha}}$ and the observed energy is expressed as: $\boldsymbol{E} = -\boldsymbol{p}.\boldsymbol{u}_{\boldsymbol{obs}}$.

Coordinate Bases

The four-velocity u is a familiar example of a vector. Given a world line $x_\alpha(\tau)$, so the components of its four-velocity is expected to be

$$u^\alpha = dx^\alpha/d\tau \quad (2.10)$$

But in order to find what basis these components are in we use the relation $d\tau^2 = -ds^2$ from relations (2.1) and (2.2) , then we get :

$$g_{\mu\nu}u^\mu u^\nu = g_{\mu\nu}\left(\frac{dx^\mu}{d\tau}\right)\left(\frac{dx^\nu}{d\tau}\right) = -1 \quad (2.11)$$

So we find that eqn. (2.10) are the components of the four-velocity in a different kind of ba: (2.12) where:

$$e_\alpha(x).e_\beta(x) = g_{\alpha\beta}(x)$$

These are the defining relations of a coordinate basis where generally we get:

$$a.b = g_{\alpha\beta}a^\alpha b^\beta \quad (2.13)$$

What is a Black Hole? (A Physical and Geometric point of view):

A black hole is an object so dense that it sufficiently bends the Space-time around it so that nothing can escape. The external outermost surface of a Black Hole form which nothing can escape is called the event horizon. From a physical point of view, we can picture black holes in terms of escape velocity. For example, on the Earth, if we were to toss a ball into the air, the overwhelming force of gravity due to the mass of the Earth would cause it to fall back to the ground. However, suppose we had a launcher that could shoot the ball at much larger velocities. As we increase the velocity, the ball will go higher before it falls back down. With a launcher powerful enough, we could even shoot the ball with such a velocity that it would leave the atmosphere of the Earth and continue on into space. The minimum velocity required for the ball to leave and not fall back to Earth is called the escape velocity.

Let us continue with a geometrical interpretation of this. Notice that for an object to be a Black-hole, it must have a sufficiently large density and not mass. Now, we could think of a space-time without mass as a large flat frictionless rubber sheet, similar to the surface of a trampoline. If we were to place a bowling ball on this surface, then the sheet would flex in the region around the ball, but would be flat everywhere else. This is analogous to adding an isolated static spherically symmetric mass into the space-time, such as a star. Suppose we place a marble adjacent to the bowling ball, and then tap it so that it rolls up the flexed region. If we tap it softly, it will roll up the curvature, but then roll back down to the bowling ball. As we tap it harder, it will roll further up the curvature before falling back until we tap it hard enough so that it reaches the flat region and continues on in a straight direction.

Like the example above, the minimum velocity of the marble for which it escapes the flexed region is called the escape velocity. Now, let us increase the density of the bowling ball by increasing its mass while leaving the size the same. In doing this, the region around the bowling ball will be deeper and the slope of the flexed region will increase. Then we will have to tap the marble harder in order for it to make it to the flat region, that is, the escape velocity

will increase. Now let us increase the mass of the bowling ball (while keeping its size fixed) so that the flexed region is so deep and the slope is so steep, that the ball can never escape.
We know that there exists a density of the bowling ball such that the marble can never escape because nothing can travel faster than the speed of light. For heuristic purposes, we can even pretend photons of light are particles with mass $\mathbf{E} = \mathbf{m}c^2$ as given by the relation from Einstein's special theory of relativity. Thus, there is a finite limit to how fast an object can go, but there is no limit to how large an escape velocity can be. So, if an object is so dense that its escape velocity is greater than the speed of light, then nothing can escape the gravity of that object and such a special compact object is called a ***black hole***.

Chapter 3

GEODESICS, CURVATURE TENSOR AND EINSTEIN FIELD EQUATIONS

Tensors for General Relativity

So far we have discussed the behavior of vectors in curved space-time and different bases. We have also described the qualitative concept of a Black Hole in the light of the geometric curvature of space-time and also from the physical angle of escape velocity. Now we shall generalize our description further by employing the language and properties of Tensors. Since we have already realized curved Space-time as the basis of General Relativity it will be natural for us to take *that* as the background space for our tensors. The coordinates are no longer required to have special physical meaning and arbitrary coordinate systems are permitted, as long as they cover the space-time smoothly (patch wise if necessary). Different such systems are related to each other by smooth and invertible transformations $\boldsymbol{x^{i'}} = \boldsymbol{x^{i'}}(\boldsymbol{x^{i}})$. When we transform coordinates, tensor components undergo their typical tensor transformations, but now these usually vary from point to point unlike the universal Lorentz transformations of Special theory. In curved space-time we can no longer picture vectors as displacements, but physicists have little trouble picturing them as scalar multiples of *differential* displacements $\boldsymbol{dx^{\mu}}$**.** In particular, two *directions* $\boldsymbol{dx}$ and $\boldsymbol{\delta x}$ are *orthogonal* if the following relation is satisfied:

$$\boldsymbol{g_{\mu\nu}\, dx^{\mu}\delta x^{\nu} = 0} \tag{3.1}$$

Since orthogonal coordinates can considerably simplify the mathematics, it is useful to know that in 2- and 3-dimensional spaces orthogonal coordinates always exist. However, in higher dimensions this is unfortunately no longer true. Only in one important respect do the 4-tensors of Special Theory of Relativity not generalize simply to General Relativity. As the partial differentiation of tensors is a tensorial operation *only* as long as the permitted coordinate transformations are linear—as they are for the Cartesian tensors of classical physics and the 4-tensors of Special Theory of Relativity. But in General theory of Relativity non-linear coordinate transformations are forced on us. And yet, neither physics nor geometry can progress without differentiation. So a more general tensorial operation had to be found: this came to be known as *covariant differentiation*. We shall now approach the topic of Geodesics via Covariant Differentiation.

Geodesics

In general relativity, gravity is formulated as a geometric interpretation, and as such, we must discard the classical Newtonian view of gravity. Instead, we can think of an object in a gravitational field as traveling along a **geodesic** in the 4-dimensional space-time. Due to this geometric interpretation, geodesics are very important in describing motion due to gravity. A **geodesic** is commonly defined as the shortest distance between two points. We are familiar with a geodesic in flat Euclidean geometry; it is simply a line between the two points. So on a sphere it will be a path along any one great circle as shown in Fig 3(a). However, as we move to 4-dimensional space-time, it is not always so simple. In calculating the geodesic on a curved surface of n dimensions of space, the curvature must be taken into account.

For instance, let us return to our example of the bowling ball on the frictionless rubber sheet. Suppose we roll a marble towards the flexed region. Assume that the marble starts on a flat part of the surface, and that it does not run into the bowling ball. Then the marble would initially roll straight toward the flexed region, but upon entering the curvature, it would appear to bend with the surface and exit the region heading straight out in a different direction. This is analogous to the deflection of a comet's trajectory by the gravitational influence of the Sun. In this instance, the marble and the comet are both following "straight" paths on the curved surface which are both geodesics.

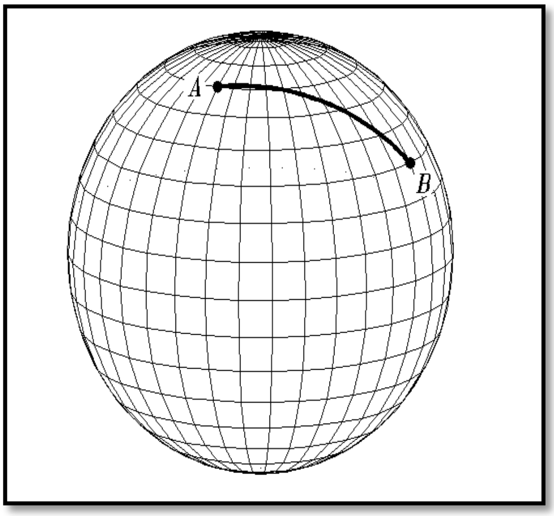


Fig. 3(a)

Now we know that the general principle for the motion of free test particles in curved space-time is the same as that for flat space-time. So using **Variational Principle** we can say that "the world line of a free test particle between two time like separated points extremizes the proper time between them."

There are only two differences from the flat-space variational principle for free particle motion when compared to that of curved space: (1) The word test has been added to the statement to make it clear that it applies the motion of bodies that are not a significant source of curvature. (2) The proper time is determined by a general metric $g_{\alpha\beta}(x)$ rather than the flat space-time metric defined through $\eta_{\mu\nu}$ as shown in relation (1.4).

Thus, in general relativity, we will deduce the equations of motion from the variational principle. Here the extremal proper time world lines are called geodesics, and the equations of motion that determine them comprise the geodesic equation.

Knowing that proper time is defined by the relation:

$$\tau_{AB} = \int_A^B \sqrt{-ds^2} = \int_A^B \sqrt{-g_{\alpha\beta} dx^\alpha dx^\beta} \tag{3.2}$$

In order to write this as an integral that we can compute, we consider a parameterized worldline, $x^\alpha = x^\alpha(\sigma)$ where the parameter $\sigma = 0$ at point A and $\sigma = 1$ at point B. then we write:

$$\tau_{AB} = \int_0^1 \left[-g_{\alpha\beta}\left(\frac{dx^\alpha}{d\sigma}\right)\left(\frac{dx^\beta}{d\sigma}\right)\right]^{\frac{1}{2}} \text{ where } d\sigma = \int_0^1 L\left[\frac{dx^\alpha}{d\sigma}, x^\alpha\right] d\sigma \tag{3.3}$$

Here we have introduced the Lagrangian, $L\left[\frac{dx^\alpha}{d\sigma}, x^\alpha\right]$

Such that

$$L = \frac{d\tau}{d\sigma}$$

Therefore, for functions$f = f(\tau(\sigma))$, we have

$$\frac{df}{d\sigma} = \frac{df}{d\tau}\frac{d\tau}{d\sigma} = L\frac{df}{d\tau}$$

We will use this later to change derivatives with respect to our arbitrary parameter $\boldsymbol{\sigma}$ to derivatives with respect to the proper time, $\boldsymbol{\tau}$**.**

Using variational methods as seen in classical dynamics, we obtain the Euler-Lagrange equations in the form

$$-\frac{d}{dx}\left(\frac{\partial L}{\partial\left(\frac{dx^\gamma}{d\sigma}\right)}\right) + \frac{\partial L}{\partial x^\gamma} = \mathbf{0} \tag{3.4}$$

Now solving each term of the Euler-Lagrange's equation, we can arrive at the **geodesic equation**, as written below:

$$\frac{d^2x^\alpha}{dr^2} = \Gamma^\alpha_{\delta\beta}\frac{dx^\delta}{dr}\frac{dx^\beta}{dr} \tag{3.5}$$

here the Christoffel symbols satisfy the following relation:

$$g_{\alpha\gamma}\,\Gamma^\alpha_{\delta\beta} = \frac{1}{2}\left[\frac{\partial g_{\gamma\delta}}{\partial x^\beta} + \frac{\partial g_{\gamma\beta}}{\partial x^\delta} - \frac{\partial g_{\delta\beta}}{\partial x^\gamma}\right] \tag{3.6}$$

This is a linear system of equations for the Christoffel symbols. If the metric is diagonal in the coordinate system, then the computation is relatively simple as there is only one term on the left side of Equation (3.5). In general, we do not need to use the metric inverse of$g_{\alpha\beta}$. Another important property is that the Christoffel symbol is symmetric in the lower indices:

$$\Gamma^\alpha_{\delta\beta} = \Gamma^\alpha_{\beta\delta}$$

We can solve for the Christoffel symbols by introducing the inverse of the metric, $g^{\mu\gamma}$; satisfying

$$g^{\mu\gamma}g_{\alpha\gamma} = \delta^\mu_\alpha \tag{3.7}$$

Here, δ^μ_α is the Kronecker delta, which vanishes for $\mu \neq \alpha$ and is 1 otherwise.

Considering that we get:

$$g^{\mu\gamma}g_{\alpha\gamma}\Gamma^\alpha_{\delta\beta} = \delta^\mu_\alpha\Gamma^\alpha_{\delta\beta} = \Gamma^\mu_{\delta\beta} \tag{3.8}$$

Therefore,

$$\Gamma^\mu_{\delta\beta} = \frac{1}{2}g^{\mu\gamma}\left[\frac{\partial g_{\gamma\delta}}{\partial x^\beta} + \frac{\partial g_{\gamma\beta}}{\partial x^\delta} - \frac{\partial g_{\delta\beta}}{\partial x^\gamma}\right] \tag{3.9}$$

The Riemann curvature tensor

We now know that space-time tells matter how to move, and matter tells space-time how to curve. We even alluded to the metric tensor $\boldsymbol{g}_{\mu\nu}$ and its role in characterizing the geometry of curved space-time. However, we have not yet described in any detail in what way matter, and specifically mass, influences the curvature of space-time. This relation will be described through the Riemann Curvature tensor followed by the Einstein Field Equations.

We will now look back to Newton's Law of Gravitation to give a brief motivation for the solution to Einstein's Field Equations as outlined by Faber (1983). We will continue to use geometrized units, that is, $c = G = 1$. Suppose a mass M is located at the origin of a 3-dimensional system (x, y, $(x, y, z)$with position vector $\vec{X} = \langle x(t), y(t), z(t) \rangle$. Let $r = \sqrt{x^2 + y^2 + z^2}$ and define $\overrightarrow{u_r} = -\frac{\vec{X}}{r}$ to be the unit radial vector, that is, a vector which points from $\vec{X}$ to the mass M at the origin. Then the force $\vec{F}$ on a particle of mass m located at $\vec{X}$ is:

$$\vec{\boldsymbol{F}} = -\frac{\boldsymbol{Mm}}{\boldsymbol{r}^2}\overrightarrow{\boldsymbol{u_r}} = \boldsymbol{m}\frac{\boldsymbol{d}^2\vec{\boldsymbol{X}}}{\boldsymbol{dt}^2}$$

where the second equality comes from Newton's second law. Such that:

$$\frac{\boldsymbol{d}^2\vec{\boldsymbol{X}}}{\boldsymbol{dt}^2} = -\frac{\boldsymbol{M}}{\boldsymbol{r}^2}\,\overrightarrow{\boldsymbol{u_r}}$$

Now let us define $\Phi = \Phi(r)$ as the potential function

$$\boldsymbol{\Phi}(\boldsymbol{r}) = -\frac{\mathbf{M}}{\mathbf{r}}$$

Now we find that using the chain rule, the result gives us:

$$\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}^i} = \frac{\partial}{\partial \boldsymbol{x}^i}\left[\left(\vec{\boldsymbol{X}}.\vec{\boldsymbol{X}}\right)^{\frac{1}{2}}\right] = \frac{\boldsymbol{2x}^i}{2\left(\vec{\boldsymbol{X}}.\vec{\boldsymbol{X}}\right)^{\frac{1}{2}}} = \frac{\boldsymbol{x}^i}{\boldsymbol{r}}$$

and

$$\frac{\partial \boldsymbol{\Phi}}{\partial \boldsymbol{x}^i} = \frac{\partial \boldsymbol{\Phi}}{\partial \boldsymbol{r}}\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}^i}$$

We may then write

$$-\boldsymbol{\nabla\Phi} = -\left(\frac{\partial \boldsymbol{\Phi}}{\partial \boldsymbol{x}}, \frac{\partial \boldsymbol{\Phi}}{\partial \boldsymbol{y}}, \frac{\partial \boldsymbol{\Phi}}{\partial \boldsymbol{z}}\right)$$

$$= -\frac{\boldsymbol{M}}{\boldsymbol{r}^2}\left(\frac{\boldsymbol{x}}{\boldsymbol{r}}, \frac{\boldsymbol{y}}{\boldsymbol{r}}, \frac{\boldsymbol{z}}{\boldsymbol{r}}\right)$$

$$= -\frac{M}{r^2}\overrightarrow{u_r}$$

$$= \frac{d^2\overrightarrow{X}}{dt^2}$$

We can then compare the individual components of the vectors above which give us:

$$\frac{d^2x^i}{dt^2} = -\frac{\partial\Phi}{\partial x^i} \tag{3.10}$$

Notice that the left hand side of the above equation looks remarkably like a term in the geodesic equation given by equation (2). Let us now write the geodesic equation as

$$\frac{d^2x^\lambda}{d\tau^2} = \Gamma^{\alpha}_{\delta\beta}\frac{dx^\delta}{dr}\frac{dx^\beta}{dr} \tag{3.11}$$

We can then make some general insights on the similarities between equations (3.10) and (3.11). For instance, equation (3.10) relies on the first partial derivatives of the potential function and equation (3.11) relies on the first partial derivatives of the components of $g_{\mu\nu}$. Then, we could imagine that the coefficients given by the metric tensor in general relativity is analogous to the gravitational potentials of Newton's theory. In a similar line of reasoning, we would want a corresponding result for Laplace's equation which describes gravitational potentials in empty space. The potential function satisfies Laplace's equation which is given by

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0 \tag{3.12}$$

In general relativity, for the analogy to hold, we would need an equation involving the second partial derivatives of the metric tensor components $g_{\mu\nu}$. Additionally, we want the equation to be invariant, so that it is independent of the coordinate system used.

From the treatment given by Faber (1983), the above requirements force our equation to be a function of $R^{\lambda}_{\mu\nu\sigma}$, which are components of the Riemann curvature tensor, and $g_{\mu\nu}$. The Riemann curvature tensor is itself a function of the metric coefficients $g_{\mu\nu}$and their first and second derivatives, and so it relies solely on the intrinsic properties of a surface. So the **Reimann curvature tensor** is finally represented by the following relation:

$$R^{\lambda}_{\mu\nu\sigma} = \frac{\partial\Gamma^{\lambda}_{\mu\sigma}}{\partial x^\nu} - \frac{\partial\Gamma^{\lambda}_{\mu\nu}}{\partial x^\sigma} + \Gamma^{\beta}_{\mu\sigma}\Gamma^{\lambda}_{\nu\beta} - \Gamma^{\beta}_{\mu\nu}\Gamma^{\lambda}_{\beta\sigma} \tag{3.13}$$

However from the physical point of view, we know that this equation is only complete if it is capable of representing the flat space-time of special relativity in one of its solutions. We can guess that the Riemann curvature tensor would be a good candidate for our equation, however, to allow for the solution of flat space-time the curvature tensor must be zero.

That is:

$$\boldsymbol{R^{\lambda}_{\mu\nu\sigma} = 0} \tag{3.14}$$

If the curvature tensor, and thus the curvature of our surface, is zero then, we will *only* have flat space-time and there will be no gravitational fields. This is too restrictive and so we need another equation that allows our space-time to have curvature. With this in mind, we will now look at Einstein's field equations which do satisfy the above requirements.

Einstein's Field Equations

We know that we would want the field equations to rely on the metric tensor components $g_{\mu\nu}$ and their first and second partial derivatives. It should relate these components, which describe the curvature of space-time, to the distribution of matter throughout space-time.

In the previous section, we saw that the Riemann curvature tensor was too restrictive. However, if we set $\sigma = \lambda$ in equation (4.4) and then sum over λ we obtain the components of the less restrictive Ricci Tensor.

Now**,** The **Ricci tensor** is obtained from the curvature tensor by summing over one index:

$$\boldsymbol{R_{\mu\nu} = R^{\lambda}_{\mu\nu\lambda} = \frac{\partial \Gamma^{\lambda}_{\mu\lambda}}{\partial x^{\nu}} - \frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\lambda}} + \Gamma^{\beta}_{\mu\lambda}\Gamma^{\lambda}_{\nu\beta} - \Gamma^{\beta}_{\mu\nu}\Gamma^{\lambda}_{\beta\lambda}} \tag{3.15}$$

Einstein's vacuum field equations for general relativity are the system of second order partial differential equations

$$\boldsymbol{R_{\mu\nu} = \frac{\partial \Gamma^{\lambda}_{\mu\lambda}}{\partial x^{\nu}} - \frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\lambda}} + \Gamma^{\beta}_{\mu\lambda}\Gamma^{\lambda}_{\nu\beta} - \Gamma^{\beta}_{\mu\nu}\Gamma^{\lambda}_{\beta\lambda} = 0} \tag{3.16}$$

where was defined in equation (3.9) as

$$\boldsymbol{\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\beta}\left[\frac{\partial g_{\mu\beta}}{\partial x^{\nu}} + \frac{\partial g_{\nu\beta}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\beta}}\right]}$$

Hence, the vacuum field equations describe space-time in the absence of mass, and so are analogous to Laplace's Equation. The field equations are a system of second order partial differential equations in the unknown function$g_{\mu\nu}$. Notice that this relates 16 equations and 16 unknown functions. The $g_{\mu\nu}$ determines the metric form of space-time and therefore all intrinsic properties of the 4-dimensional semi-Riemannian manifold that is space-time, including curvature. Now if we take into consideration the Hilbert action and solve by varying the Riemann Curvature Tensor with respect to the metric tensor, then for the variation of the Hilbert action for which the extremum is zero we get the Einstein Field equation as:

$$\frac{c^4}{16\pi G}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) - \frac{1}{2}T_{\mu\nu} = 0$$

Now, simplifying the above relation we finally obtain the form :

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{3.17}$$

where $\mathbf{R_{\mu\nu}}$ is the Ricci curvature tensor,

R the scalar curvature,

$g_{\mu\nu}$ the metric tensor,

G is Newton's gravitational constant, **c** the speed of light in vacuum,

$T_{\mu\nu}$ the stress–energy tensor.

However in the case in which the energy-momentum tensor $T_{\mu\nu}$ is zero in the region under consideration, then the field equations are also referred to as the **vacuum field equations**. By setting $T_{\mu\nu} = 0$ the vacuum equations can be written as:

$$R_{\mu\nu} = 0 \tag{3.18}$$

The solutions to the vacuum field equations are called vacuum solutions. Flat Minkowski space is the simplest example of a vacuum solution. More importantly to serve our intended purpose we shall consider an important nontrivial solution to the Vacuum Field Equations which is called the **Schwarzschild solution**. (the derivations of Einstein's Vacuum Field Equations and complete Field Equations are done in detail from the Principle of Action in the Appendix section)

Chapter 4
THE SCHWARZSCHILD METRIC

Arriving at the Metric

We start by considering first the Minkowski interval $\boldsymbol{ds^2}$ followed by a few subsequent steps which lead to the formation of the Schwarzschild coordinates and the Schwarzschild metric which will be done in complete detail in the appendix section of the project. Now, we begin with:

$$\boldsymbol{ds^2 = c^2 dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)}$$

The term in brackets expresses spherical symmetry or isotropy (no preference for any direction). Any spherically symmetric metric must have a term of this form. Thus a general isotropic metric can be written

$$\boldsymbol{ds^2 = Adt^2 - Bdt\,dr - Cdr^2 - D(d\theta^2 + \sin^2 d\phi^2)} \tag{4.1}$$

- Expect symmetry under $\phi \rightarrow -\phi, \theta \rightarrow \pi - \theta$ so no cross terms with$dr\; d\theta$ or $d\theta\; dt$.
- A, B, C and D cannot depend on θ or ϕ otherwise isotropy is broken $\Rightarrow$ functions of r and t only.

We can define a new radial coordinate r' such that $(r')^2 = D$, and so the metric becomes:

$$\boldsymbol{ds^2 = A'dt^2 - B'dtdr' - C'(dr')^2 - (r')^2(d\theta^2 + \sin^2\theta\, d\phi^2)} \tag{4.2}$$

The above metric is still general. So, dropping the primes, with this radial coordinate, the area of a sphere is still$4\pi r^2$, but r is not necessarily the ruler distance from the origin.

Finally we can transform the time coordinate using

$$\boldsymbol{dt = f\, dt' + g\, dr},$$

Choosing f and g such that dt is an exact differential and so that the cross terms in $dr\, dt'$ cancel. We are left with

$$\boldsymbol{ds^2 = A(r,t)dt^2 - B(r,t)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)} \tag{4.3}$$

as the general form of an isotropic metric.

We specialize further by looking for time-independent metrics, like:

$$\boldsymbol{ds^2 = A(r)dt^2 - B(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)}$$

This is also static as it is invariant under the transformation $\boldsymbol{t} \rightarrow \boldsymbol{-t}$.

Now, we will find the metric around a star such as the Sun, i.e. in empty space where $\boldsymbol{T_{\alpha\beta} = 0}$ and $= \boldsymbol{T^{\alpha}_{\alpha} = 0} \Rightarrow \boldsymbol{R = 0}$, so the field equations:

$$\left(\boldsymbol{R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}}\right) = -\frac{\boldsymbol{8\pi G}}{\boldsymbol{c^4}} \boldsymbol{T_{\alpha\beta}},$$

reduce to

$$\boldsymbol{R_{\alpha\beta} = 0}$$

where $\boldsymbol{R_{\alpha\beta}}$ comes from

$$\boldsymbol{R_{\alpha\beta} = \Gamma^{\rho}_{\alpha\beta,\rho} - \Gamma^{\rho}_{\alpha\rho,\beta} + \Gamma^{\sigma}_{\alpha\beta}\Gamma^{\rho}_{\sigma\rho} - \Gamma^{\sigma}_{\alpha\rho}\Gamma^{\rho}_{\sigma\beta}}$$

While

$$\boldsymbol{\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta}(g_{\delta\gamma,\beta} + g_{\beta\delta,\gamma} - g_{\beta\gamma,\delta})}$$

Working out Γ then R followed by much algebra leads to coupled, ordinary differential equations for A and B and one gets the following relation :

$$\boldsymbol{A(r) = \alpha\left(1 + \frac{k}{r}\right)}, \tag{4.4}$$

$$\boldsymbol{B(r) = \alpha\left(1 + \frac{k}{r}\right)^{-1}},$$

Where α and k are constants.

Now by making the following replacements that is: $\alpha = \boldsymbol{c^2}$ and $k = -\frac{\boldsymbol{2GM}}{\boldsymbol{c^2}}$**.** We arrive at the **Schwarzschild metric**:

$$\boldsymbol{ds^2 = c^2\left(1 - \frac{2GM}{c^2 r}\right) dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)} \tag{4.5}$$

This applies outside a spherically-symmetric object, e.g. for motions of the planets but not inside the Sun. Schwarzschild's solution is important as the first exact solution of the field equations. In geometrized units the Schwarzschild line element has the form:

$$\boldsymbol{ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)} \tag{4.6}$$

The coordinates above are called Schwarzschild coordinates and the corresponding metric $g_{ab}(x)$ is called Schwarzschild metric. Explicitly the metric $g_{ab}$is

$$g_{ab} = \begin{bmatrix} 1-\frac{2GM}{r} & 0 & 0 & 0 \\ 0 & -\left(1-\frac{2GM}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{bmatrix}. \quad (4.7)$$

Birkhoff's theorem

If one does not impose time-independence, i.e. $A = A(r,t), B = B(r,t)$, and solves $R_{\alpha\beta} = 0$, one still finds Schwarzschild's solution (Birkhoff 1923), i.e.The geometry outside a spherically symmetric distribution of matter is the Schwarzschild geometry.

This means spherically symmetric explosions cannot emit gravitational waves.

It also means that space-time inside a hollow spherical shell is flat since it must be Schwarzschild-like but have $M = 0$. Flat implies no gravity, the GR equivalent of Newton's "iron sphere" theorem. Used in semi-Newtonian justifications of the Friedman equations.

Properties of Schwarzschild Metric

- Time Independent: The metric is independent of t. There is a Killing Vector ξ associated with this symmetry under displacements in the coordinate time t, which has the components
$$\boldsymbol{\xi^\alpha = (1, 0, 0, 0)}$$
- Spherically Symmetric: The geometry of a two-dimensional surface of constant t and constant r in the four dimensional geometry is summarized by the line element
$$\boldsymbol{d\Sigma^2 = r^2(d\theta^2 + \sin^2\theta\, d\phi^2)}$$
(4.8)
This describes the geometry of a sphere of radius r in flat three-dimensional space. The Schwarzschild geometry thus has the symmetries of a sphere with regard to changes in the angles θ and ϕ. The killing vector associated with this symmetry is
$$\boldsymbol{\eta^\alpha = (0, 0, 0, 1)} \quad (4.9)$$
- Schwarzschild radius: $\boldsymbol{r = \frac{2GM}{c^2}}$ is called the Schwarzschild radius and is the characteristic length scale for curvature in Schwarzschild geometry.

5. Gravitational Collapse and Black Holes

The Schwarzschild Black Hole using Eddington-Finkelstein Coordinates

To get at the essential physics of gravitational collapse, let's consider the idealized case where the collapsing body and the space-time outside it are spherically symmetric. The geometry outside a spherically symmetric gravitational collapse is the time-independent Schwarzschild geometry already explored before.

We now have to face up to the singularities in the Schwarzschild metric at the radii $\boldsymbol{r = 2M}$ an**d** $\boldsymbol{r = 0}$ and the significance of the change in sign of g_{tt} and g_{rr} **at** $\boldsymbol{r = 2M}$. This section discusses the properties of the Schwarzschild geometry all the way down to $\boldsymbol{r = 0}$ without including the collapsing matter.

The singularity in the Schwarzschild metric at $\boldsymbol{r = 2M}$ turns out not to be singularity in the geometry of spacetime, but a singularity in Schwarzschild coordinates. It is a coordinate singularity in the sense discussed earlier. To show this, it is only necessary to exhibit one coordinate system in which the metric is not singular at $r = 2M$. There are many, but Eddington-Finkelstein coordinates are an especially simple example. Using these coordinates we will be able to understand why the Schwarzschild geometry is a black hole.

To introduce Eddington-Finkelstein coordinates, begin with Schwarzschild coordinates (t, r, θ, ϕ), in which the metric is summarized as before, and trade the Schwarzschild time coordinate t for a new coordinate v defined by

$$\boldsymbol{t = v - r - 2M \log\left|\frac{r}{2M} - 1\right|} \tag{5.1}$$

Starting from either $\boldsymbol{r} < 2\boldsymbol{M}$ or $\boldsymbol{r} > 2\boldsymbol{M}$ and transforming t to v in the line element discussed earlier, gives:

$$\boldsymbol{ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dv\, dr\ - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)}$$

(5.2)

We must take into account however that this is not a new geometry! It's the same time-independent, spherically symmetric geometry represented by the Schwarzschild metric, but with a different system of coordinates for labeling the points.

The absence of any singularity at $\boldsymbol{r = 2M}$ in (5.2) shows that the singularity there in Schwarzschild coordinates is just a coordinate singularity. The line element (5.2) is fit for describing physics outside, at, and inside the Schwarzschild radius. Its nonsingular character shows that observers falling through the radius $\boldsymbol{r = 2M}$ will see nothing special about the local space-time. Eddington-Finkelstein coordinates are therefore useful for the study of ongoing gravitational collapse.

Contrast the situation at $\boldsymbol{r = 2M}$ with that at $\boldsymbol{r = O}$. There the metric is singular in both the Schwarzschild and Eddington-Finkelstein coordinate systems. As we will see quantitatively**,** $\boldsymbol{r = 0}$ is a place of infinite spacetime curvature and infinite gravitational forces **— a real physical singularity**. Observers falling into $r = 0$ will definitely see something special about the local space-time. They will be destroyed and will be sucked into the singularity or resultant Black-Hole!

Light Cones of the Schwarzschild Geometry

The key to understanding the Schwarzschild geometry as a black hole is the behavior of radial light rays. These move along world lines for which $d\theta = d\phi = 0$ (radial) and $ds^2 = 0$ (null), i.e., from (5.2), those for which

$$\boldsymbol{-\left(1 - \frac{2M}{r}\right) dv^2 + 2dv\, dr = 0}$$

(5.3)

An immediate consequence of this is that some radial light rays move along the curves

$\boldsymbol{v = constant}$ **(Ingoing radial light rays)**

From (5.1) we see that these are ingoing light rays because as t increases, r must decrease to keep v constant. The other possible solution to (5.3) is

$$\boldsymbol{-\left(1 - \frac{2M}{r}\right) dv + 2\, dr = 0}$$

(5.4)

This can be solved for $\frac{dv}{dr}$ and the result integrated to find that these radial light rays move on the curves

$$\boldsymbol{v - 2\left(r + 2M \log\left|\frac{r}{2M} - 1\right|\right) = constant}$$

Radial light rays (5.5)
Outgoing $\boldsymbol{r > 2M}$
Ingoing $\boldsymbol{r < 2M}$

When one of these light rays is far from the black hole, it is *outgoing* because (5.5) becomes $t = r + constant$ as (5.1) shows. But when $r < 2M$, these light rays are *ingoing* because r decreases as v increases.

As shown in Fig (5.1(a)) below, the region outside $r = 2M$ from which light can escape to

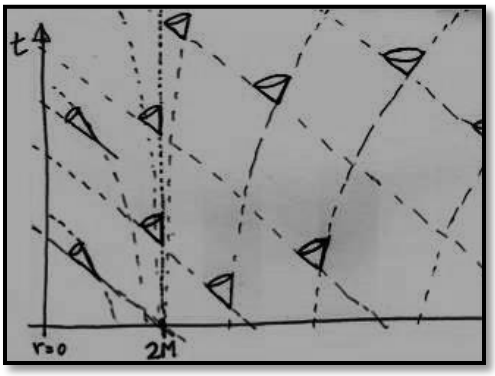


Fig.(5.1(a))

infinity and the region inside$r = 2M$, where gravity is so strong that not even light can escape. This is the defining feature of a *black hole* geometry. The surface $r = 2M$ is called the *event horizon* (or, often more briefly, just the *horizon*) of the black hole.

Now, let us briefly look into the geometry of the horizon and singularity of the black hole under consideration. The horizon $\boldsymbol{r = 2M}$ is a three-dimensional null surface in space-time of the kind discussed generally earlier. Its normal vector points in the r-direction and is a null vector. Like the future null cone in flat space, the horizon has a one-way property-once crossed it is not possible to cross back. However, unlike the light cones of flat space, the horizon is stationary, not expanding. The horizon is generated by those radial light rays that neither fall into the singularity nor escape to infinity.

Collapse to a Black Hole

The radially moving particles at the surface of the collapsing star follow time-like world lines that lie inside the light cone at each point of space-time they pass through, just like any other particle. The world line of the surface of a collapsing ball of pressureless matter that starts from rest at infinity provides a simple instance that is illustrated in Figures 5(a) and 5(b).Outside the collapsing surface, the geometry of spherically symmetric collapse is the Schwarzschild geometry, including the horizon after the star has crossed the Schwarzschild radius $r = 2M$ and the singularity after it hits $r = O$. Inside the surface (the heavily shaded region in Figure 5(a)) the geometry is different, dependent on the detailed properties of the matter, but matching Schwarzschild geometry at the surface. In the following page we shall discuss the point of view of two observers in the spherical geometry of a collapsing star and the formation of a Black Hole along with suitable diagrams and illustrations:

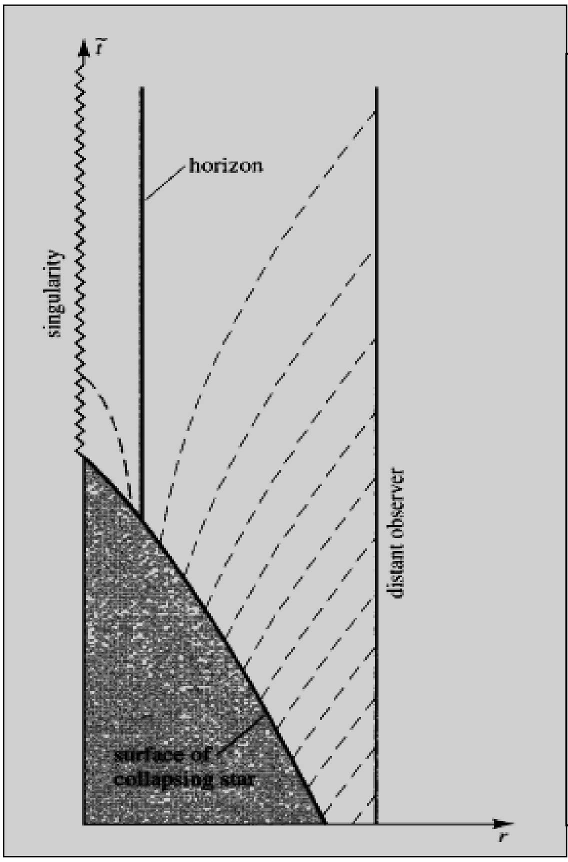


Figure 5(a) The story of two observers in the geometry of a collapsing spherical star:
One observer stays at a fixed Schwarzschild radius r_R outside the star. The other follows its surface to smaller and smaller radii, sending out light signals at equal proper time intervals according to a clock falling with the surface. These light signals propagate out to the distant observer along the dotted curves shown. Only light rays emitted before the radius $\boldsymbol{r = 2M}$ is crossed reach the distant observer. The distant observer, therefore, never sees the surface of the star cross $\boldsymbol{r = 2M}$. The pulses arrive separated by longer and longer intervals as measured by the distant observer's clock. The light from the falling star becomes dimmer and dimmer and increasingly red-shifted. A black hole is formed. Only the part of this Eddington-Finkelstein diagram outside the surface of the collapsing star (not heavily shaded) is meaningful. At the surface, the geometry matches the geometry inside the star, which is not the Schwarzschild geometry.

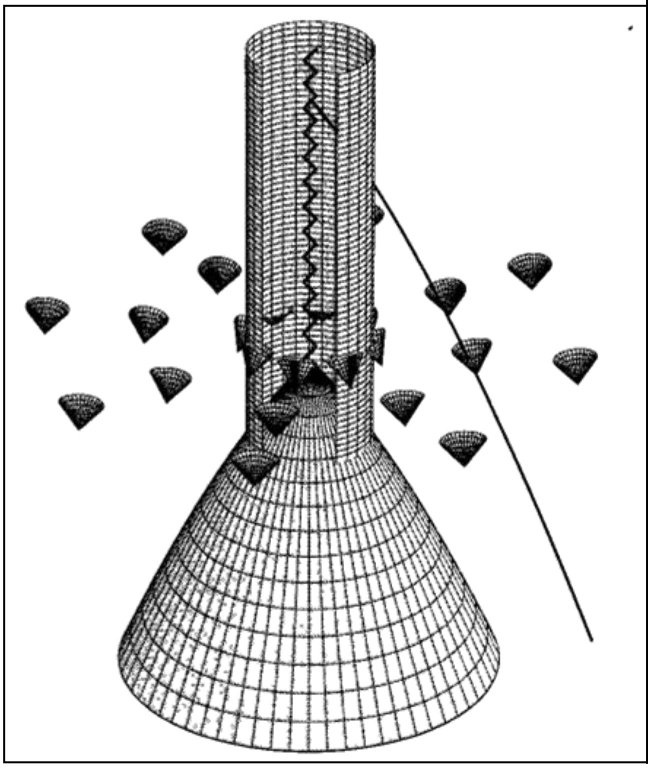


Figure 5(b) The formation of a black hole
Some essential features of a spherically symmetric gravitational collapse that forms a black hole are shown in this three- dimensional space-time diagram:
Eddington-Finkelstein coordinates $(\boldsymbol{\tilde{t} \equiv v - r, r, \phi})$ are used as cylindrical coordinates to label points in the diagram — $\tilde{t}$ vertically, r as radius from the axis of symmetry, and ϕ as azimuthal angle about that axis. The bottom surface is the world sheet swept out by the surface of the collapsing star as it progresses to smaller and smaller radii and eventually to a singularity at $\boldsymbol{r = 0}$. The vertical cylinder is the horizon at the Schwarzschild radius $\boldsymbol{r = 2M}$. The horizon conceals the singularity from any distant observer but has been cut away in the illustration to reveal it. The world line of an observer falling freely from rest at infinity through the horizon and into the singularity is shown. The orientation of the future light cones at different radii on one $\tilde{t} = constant$ surface is shown. These tip more and more toward the center as they get closer to it.

Thus, once across the Schwarzschild radius $r = 2M$, gravitational collapse to a singularity is the inevitable fate of the star. No new source of pressure at high densities can save it from collapse to zero size and infinite density. As long as the collapse remains spherical, the surface must travel *some* time-like radial world line, and *all* of these lead to the singularity at $r = 0$. Even if the star becomes non-spherical inside the horizon, it turns out that collapse to a singularity is inevitable. For the observer riding down with the star, there is also no way to escape destruction in the singularity once across the radius $r = 2M$.

Non-spherical Gravitational Collapse

We know that the realistic collapse of stars cannot be completely spherical as spherical collapse is only an ideal case. So let us try to analyze theoretically how much of this picture of spherical collapse persists in a realistic case by considering the following topics under the light of Non-Spherical gravitational Collapse:

- ***Formation of a Singularity*:** As we have seen, once the surface of a spherical collapsing star crosses the Schwarzschild radius, $r = 2M$, gravitational collapse to a singularity is inevitable. The geometry allows no escape from the region inside the horizon or for the collapse to stop. The formation of a singularity in spherical gravitational collapse is a specific illustration of the singularity theorems of general relativity. Roughly speaking, these theorems show that any gravitational collapse that proceeds far enough results in a singularity in space-time geometry. The singularity formed in spherical collapse is thus not an artifact of the special symmetry but a feature of more general collapse situations.

- ***Formation of an Event Horizon:*** The singularity formed in spherical collapse is inside the horizon, hidden from observers outside. The fact that it is hidden is important, because a singularity is a place where the predictive power of the theory breaks down, but information about this breakdown can never reach observers outside.

- ***Area Increase*:** The area of a black hole increases when mass falls into it in a spherically symmetric way. However, even if mass falls in a non-spherically symmetric fashion, the area of the horizon still increases. That is a consequence of the area increase theorem for black holes. This behavior of the area of a black hole recalls the increase in entropy in thermodynamics.

Chapter 6

KRUSKAL EXTENSION AND PENROSE DIAGRAMS

Kruskal-Szekers Coordinates

To understand the complete structure of space-time for $r \leq 2GM$, we shall now introduce the Kruskal-Szekers coordinate system which does away with all of the problems of the metric being ill-defined at various points. Now we know that the Schwarzschild metric is 4-dimensional metric. As it has a spherical symmetry we can treat it as 2-dimensional metric. So, we will only consider ***'r –t'*** part and discuss the singularity at $r = 2M$ of the Schwarzschild metric. So, we can write the metric as:

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} \tag{6.1}$$

Where, $-\infty < t < \infty$,$0 \leq r < \infty$. Now, we will consider null geodesic. In null condition we can write:

$$ds^2 = g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = 0 \tag{6.2}$$

Comparing it with equation (6.1), we get,

$$-\left(1 - \frac{2M}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 = 0 \tag{6.3}$$

Solving equation (6.3) and integrating it we get:

$$t = \pm(r + 2M \ln\left(\frac{r}{2M} - 1\right)) + const. \tag{6.4}$$

Henceforth we can define;

$$r* = r + 2M \ln\left(\frac{r}{2M} - 1\right) \tag{6.5}$$

r∗ is known as 'Regge-Wheeler-Tortoise' coordinate. So, we write eqn(6.4) as:

$$t = \pm r* + const. \tag{6.6}$$

So we get:

$$\left(\frac{dr*}{dr}\right) = \left(1 - \frac{2M}{r}\right)^{-1} \tag{6.7}$$

Now if we define the null coordinates as:

$$u = t - r* = u - r - 2M\, ln\left(\frac{r}{2M} - 1\right) \tag{6.8}$$

And,

$$v = t + r* = v + r + 2M \ln\left(\frac{r}{2M} - 1\right) \tag{6.9}$$

Then in this new coordinate, the Schwarzschild metric becomes:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dudv \tag{6.10}$$

But in this new coordinate we still have a singularity at r= 2M; so let us write:

$$r* = r + 2M\, ln\left(\frac{r}{2M} - 1\right) = \frac{v - u}{2} \tag{6.11}$$

Rewriting the metric from the last equation and multiplying $\left(\frac{2M}{r}\right)$ on both sides, we get the metric in the following form:

$$ds^2 = -\frac{2Me^{-\left(\frac{r}{2M}\right)}}{r} e^{\left(\frac{v-u}{4M}\right)} dudv \tag{6.12}$$

Now, let us define another set of new coordinates such that:

$$U = -e^{\frac{u}{4M}} \; and \; V = e^{\frac{v}{4M}}$$

Then; $du = 4Me^{\left(-\frac{u}{4M}\right)} dU$ and $dv = 4Me^{\frac{v}{4M}} dV$; so putting these values of *du* and *dv* in equation (6.12) we get the final form of the metric as:

$$ds^2 = -\frac{32M^3 e^{-\left(\frac{r}{2M}\right)}}{r} dU\, dV \tag{6.13}$$

Now, this metric is not singular at $r \to 2M$ so it does not blow up. In this new coordinate system, the values of **r** correspond to ***U=0*** and ***V=0***, thus this metric is no longer singular at this value. So, we have effectively extended the Schwarzschild metric to take U and V all other values for *r>0.r=0* is the physical singular point. This singularity cannot be removed by transforming coordinates. The scalar curvature $R_{abcd}R^{abcd}$ blows up at $r = 0$ and becomes infinite.

So let us now define the final transformation as:

$$T = \frac{U + V}{2} \; and \; X = \frac{U - V}{2}$$

We may also write:

$$dUdV = \frac{1}{4}\left(2\, dUdV + 2dUdV\right)$$

$$dUdV = \left(\frac{U + V}{2}\right)^2 - \left(\frac{U - V}{2}\right)^2$$

So, $$dUdV = dT^2 - dX^2$$

Therefore if we finally put these above values in equation (6.13) we get the final form of the metric that is known as the Kruskal-Szekers metric, which is:

$$ds^2 = \frac{32M^3 e^{-\left(\frac{r}{2M}\right)}}{r}\left(-dT^2 + dX^2\right) \tag{6.14}$$

In 4-dimensions the Kruskal metric is written as:

$$\boxed{ds^2 = \frac{32M^3 e^{-\left(\frac{r}{2M}\right)}}{r}\left(-dT^2 + dX^2\right) + r^2\left(d\theta^2 + sin^2\theta\, d\phi^2\right)} \tag{6.15}$$

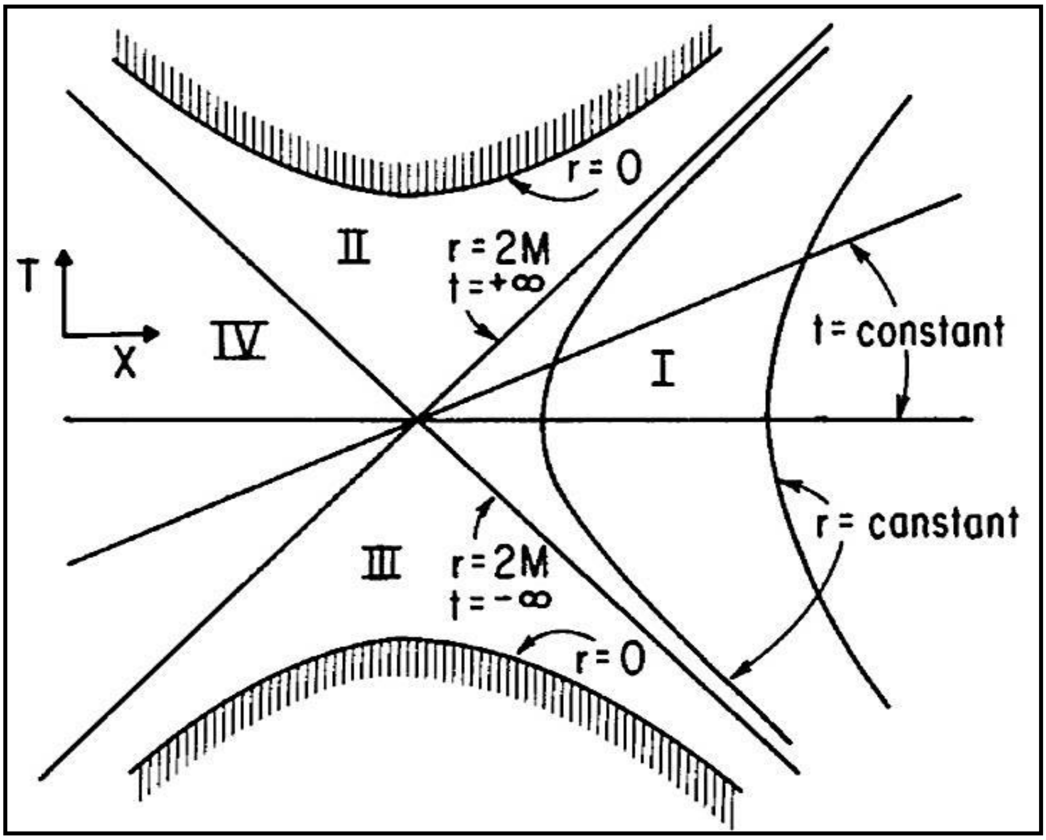
So the relation between the old and new coordinates can be shown as:

$$\left(\frac{r}{2M} - 1\right) e^{\frac{r}{2M}} = X^2 - T^2 \tag{6.16}$$

A Space-time diagram has been drawn below which represents the causal structure of the Schwarzschild metric. The null lines are 45 degrees in Kruskal coordinates. With the help of this diagram we shall now analyse and discuss the Kruskal extension of the Schwarzschild metric and its physical implications.

Geometry of the Extended Metric

Figure 6 (a)



Now let us discuss the singular points and various regions depicted in the above diagram coined as **Figure 6(a).** In the original Schwarzschild metric, we have seen that r=0 is singular point. So let us substitute r = 0 in the extended metric. Doing so we get,

$\boldsymbol{X2 - T2 = -1}$ (6.16).This is a hyperbolic curve, shown in the figure. There is physical singularity at r = 0 in the original metric. In the extended metric, we see there is still singularity at $\boldsymbol{X = \pm(T2 - 1)1/2}$. r = 0 has space like character here and exists in the region II and region III.

In the original Schwarzschild metric, $\boldsymbol{r = 2M}$ is a singular point. However, in the extended metric, it is no-longer a singular point. Thus, we can safely conclude that it is actually a coordinate singular point. $\boldsymbol{r = 2M}$ is actually corresponds to null rays in the extended space-time diagram. For, $\boldsymbol{r = 2M}$, we get from equation (6.16)

$$\boldsymbol{X^2 = T^2}$$

Or,

$$\boldsymbol{X = \pm T}$$

For this, these null lines are labelled as $\boldsymbol{t = \pm\infty}$.$\boldsymbol{r = 2M}$ shows no bad behaviour in the Kruskal coordinate. In the region $\boldsymbol{r < 2M}$ the metric is given by,

$$\boldsymbol{ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)}} \tag{6.17}$$

So, in this region behaviour of space and time gets flipped. The space component becomes 'time-like' and the time component becomes 'space-like'. Thus the metric is no longer static here. Here |x|<|t|, hence it is space like region.

r >2M is the region I in the diagram. This region represents the exterior gravitational field of a spherical body. The metric is static in this region. In this region, |x|>|t| So, it is a time-like region. Region I is the original space-time which is observable by physical instruments. It is our world. Radial in-falling matter crosses the hyperbolae and finally hits the line X= T where it crosses the horizon. Let us consider, a particle moving radially in the region I happens to cross the null line $X = T$ and region II. Once it enters this region, it can never escape from it. After some finite time, it will always fall into the singularity. If it sends a signal before falling into the singularity, then the signal will also fall into singularity. Thus, this region is called Black hole. Everything in this region is restricted within the light cone with an angle of 45 degrees and thus nothing, not even light can escape from region II. Region III is the time reversal of region II. An observer present in III must have been originated in the singularity and must leave region III again to region I. In the sixties some astronomers speculated that Quasars might be fuelled by white holes. However, observations at high resolution have unambiguously shown that the intense emission is due to matter which moves to the Black Hole and finally vanishes there. Besides these observational evidences the existence of white holes would cause severe thermodynamic problems. Thus Region III has identical properties as region II but time is reversed. Any particle or observer in this region must have originated from singularity at $\boldsymbol{X} = -(\boldsymbol{T2} - \mathbf{1})^{\frac{1}{2}}$, and will leave this region after some finite time. This region is called **White Hole**. The region III has identical properties as region I. Observer at region I cannot communicate with the region III. If a signal is sent from the region I, the signal will go to black hole. Region IV has properties identical with those of region I, and represents an asymtotically flat region which lies inside of the radius $= r_s$. Region IV and III may be unphysical solution but region II has great physical importance because this is the Schwarzschild black hole region.

Conformal Transformation and Penrose Diagrams

The causal structure of General Relativity is best discussed with the help of Penrose Diagrams which allow us to consider the respective geometry in a compactified form. In order to be able to draw space-time diagrams that capture the entire set of global properties along with the causal structure of symmetric space-time; we essentially require a conformal transformation which would bring the entire structure of infinite space onto a compact region.

A Conformal Transformation is a type of transformation which preserves the geometric angles and has a domain and range in the complex plane. It is described in terms of a Jacobian derivative matrix of coordinate transformation. For a conformal transformation, the Jacobian matrix of transformation everywhere is scalar times the rotation matrix.

A Jacobian Matrix is a matrix of all first order partial derivatives of a vector valued function. So if F is a real valued function which takes as input 'n' real elements and produces as output 'm' real elements. Then the partial derivatives of all these functions with respect to variables $\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots, \boldsymbol{x_n}$ can be organized in an m×n matrix such that the Jacobian matrix J of F is epresented as:

$$\boldsymbol{J} = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} \end{bmatrix} \tag{6.18}$$

Thus, a suitable representation with the help of conformal transformations, such that we can fit space- time along with its infinities within a finite 2- dimensional diagram is known as a Penrose Diagram.

Now, in case of choosing an appropriate function that would maintain conformity as well as take up a range of infinite values; we shall either have to go with the exponential function of the tangent function. In this case the best representation of Penrose Diagrams is possible if we either choose the tan inverse function or the tan hyperbolic function. In both the cases we see that as the value of 'x' tends towards infinity, the functions $\tan^{-1}(x)$ or $\tanh(x)$ tends towards a finite number. That is 1.57 or $\frac{\pi}{2}$ in case of $\tan^{-1}(x)$ and 1 in case of $\tanh(x)$. This can be shown graphically as follows.

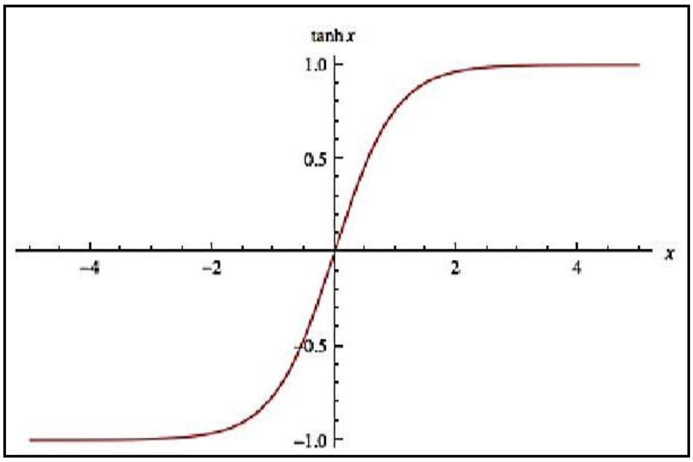


Figure 6 (b)

Figure 6 (b) shows a plot of the function $\boldsymbol{tanh}(\boldsymbol{x})$ which approaches +1 for ∞ →x and -1 for →−∞ x. This function is thus appropriate to meet our main criteria for making conformal transformations, i.e. Preserving the light cone and mapping the entire 4 dimensional infinite space on a finite portion of a 2 dimensional plane.

Now let us first consider the Minkowski space metric in polar coordinates:

$\boldsymbol{ds^2 = -dt^2 + dr^2 + r^2\ (d\theta^2 + sin^2\theta\ d\phi^2)}$; Where the ranges of time-like and space-like coordinates are : $-\infty < t < \infty;\ 0 \le r < \infty$. Now in order to get coordinates with infinite ranges, we can choose null coordinates such that: $\boldsymbol{u = t - r, v = t + r}$; with corresponding ranges $-\infty < u < \infty; -\infty < v < \infty;\ v \le u$.So the Minkowski metric in the null coordinates is given by : $\boldsymbol{ds^2 = -\frac{1}{2}\ (dudv + dvdu) + \frac{1}{4}(v-u)^2 r^2\ (d\theta^2 + sin^2\theta\ d\phi^2)}$. For the sake of simplifying calculations, we can also choose $\boldsymbol{u = \frac{1}{2}(t + r) and\ v = \frac{1}{2}(t - r)}$; where v and u have the same range as specified before. Then in this case the metric in flat space, represented in null coordinates becomes: $\boldsymbol{ds^2 = -2\ (dudv + dvdu) + (v-u)^2 r^2\ (d\theta^2 + sin^2\theta\ d\phi^2)}$. Now in order to bring infinity to a finite coordinate we can choose either the inverse tangent function or the tan hyperbolic function as shown before. So let us choose:

$$\boldsymbol{U = tan^{-1}(u)\ and\ V = tan^{-1}(v)} \tag{6.19}$$

The ranges for these coordinates are: $\boldsymbol{-\frac{\pi}{2} < U < +\frac{\pi}{2}\ \ and\ \ -\frac{\pi}{2} < V < +\frac{\pi}{2}}$; where $V \le U$.

From this we get: $\boldsymbol{dU = \frac{du}{1+u^2}}$ and since $\boldsymbol{cosu = \frac{1}{(1+u^2)^{\frac{1}{2}}}}$

$$\text{So, }\boldsymbol{cos^2u = \frac{1}{1+u^2}}$$

$$\text{Or, }\boldsymbol{dU = \frac{du}{cos^2\left(tan^{-1}(u)\right)}};\ \text{where }\boldsymbol{tan^{-1}(u) = U}$$

(6.20)

Now solving for the $(v-u)^2$ term, we get: $(\tan U - \tan V)^2 = \frac{1}{\cos^2 U \cos^2 V} \sin^2(U-V)$

Therefore, the metric becomes:

$$ds^2 = \frac{1}{\cos^2 U \cos^2 V} [-2(dUdV + dVdU) + \sin^2(U-V) r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \quad (6.21)$$

The metric in equation (6.21) is conformal in nature, but it has two null coordinates and two space-like coordinates. So we can improve the situation for ourselves by transforming to a form with one time-like and three space-like coordinates. This can be achieved by defining

$$\tau = U + V \quad and \quad \chi = U - V \quad (6.22)$$

with ranges $-\pi < \tau < +\pi, 0 \le \chi < +\pi$. The metric now is

$$ds^2 = \frac{1}{\omega^2}(-d\tau^2 + d\chi^2 + \sin^2 \chi d\Omega^2) \quad (6.23)$$

$$where \; d\Omega^2 = r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$$

$$\text{And } \omega = \cos U \cos V = \frac{1}{2}(\cos\tau + \cos\chi),$$

This can also be written in a better looking form as:

$$d\bar{s}^2 = \omega^2 ds^2 \quad (6.24)$$

$$= -d\tau^2 + d\chi^2 + \sin^2 \chi \, d\Omega^2$$

So we see that this is a successful conformal transformation as it is just a scalar times the original metric. Now we can represent this new metric describing Minkowski space in terms of a Penrose Diagram as shown below.

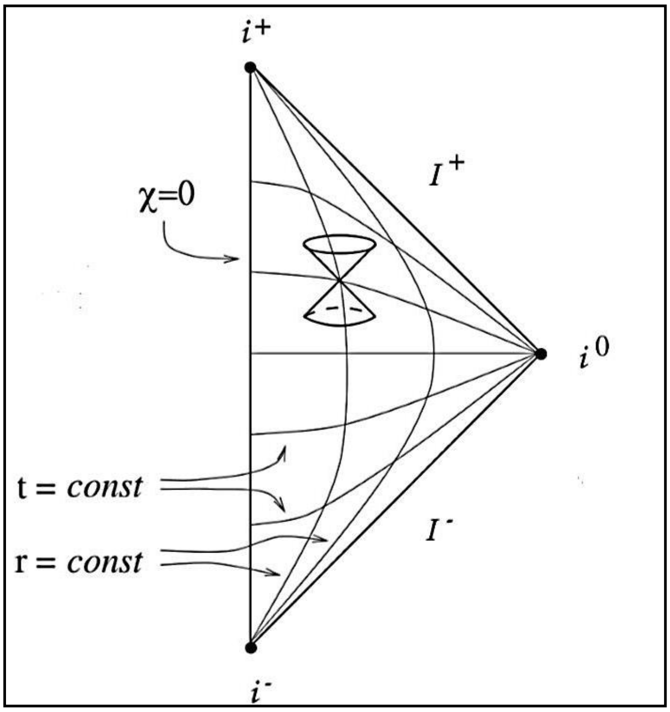


Figure 6 (c)

Figure 6(c)shows a Penrose Diagram for the Minkowski metric. Here, each point represents a 2-sphere except points at $\boldsymbol{i^0}, \boldsymbol{i^{\pm}}$. This can be seen as $\boldsymbol{sin^2\,\chi\ is\ 0\ at\ \chi\ =\ 0, \pi}$, which is to say that the radius of the sphere at that point becomes 0. So, $\boldsymbol{i^0}, \boldsymbol{i^{\pm}}$ are indeed real points. We can divide the conformal infinities of the Penrose diagram into different regions as follows:

$\boldsymbol{i^+}$ = future timelike infinity ($\tau = \pi, \chi = 0$)

$\boldsymbol{i^0}$ = spatial infinity ($\tau = 0, \chi = \pi$)

$\boldsymbol{i^-}$ = past timelike infinity ($\tau = -\pi, \chi = 0$)

$\boldsymbol{I^+}$ = future null infinity ($\tau = \pi - \chi$, $0 < \chi < \pi$)

$\boldsymbol{I^-}$ = past null infinity ($\tau = -\pi + \chi$, $0 < \chi < \pi$)

There are a few important points to note about the diagram. The radial null geodesics are at $\pm 45^o$ in the diagram. All timelike geodesics begin at $\boldsymbol{i^-}$ and end at $\boldsymbol{i^+}$, all null geodesics begin at $\boldsymbol{I^-}$ and end at I^+ and all spacelike geodesics begin and end at $\boldsymbol{i^0}$. Another point to note is that timelike curves which ends at null infinity are possible if they become asymptotically null (i.e. constant acceleration curves).So, we have indeed fit all of the Minkowski space-time on a small piece of paper.

Penrose Diagram of Schwarzschild Black Hole

In this case, we start with the null version of the Kruskal coordinates in which the metric takes the form:

$$\boldsymbol{ds^2 = \frac{16M^3 e^{-\left(\frac{r}{2M}\right)}}{r}\,(-du'dv' + dv'du') + r^2(d\theta^2 + sin^2\theta\ d\phi^2)} \tag{6.25}$$

Where: $\boldsymbol{u'v' = \left(\frac{r}{2M} - 1\right) e^{\frac{r}{2M}}}$

Then we use the same kind of transformation as we had done for Minkowski space in order to bring infinity to finite coordinate values using:

$$\boldsymbol{u'' = tan^{-1}\left(\frac{u}{(2M)^{\frac{1}{2}}}\right)\ and\ \ v'' = tan^{-1}\left(\frac{v}{(2M)^{\frac{1}{2}}}\right)} \tag{6.26}$$

With ranges: $\boldsymbol{-\frac{\pi}{2} < u'' < +\frac{\pi}{2};\ \frac{\pi}{2} < v'' < +\frac{\pi}{2};\ -\pi < u'' + v'' < \pi}$

The (u'', v'') part of the metric that is at constant angular coordinates is now conformally related to Minkowski space. In the new coordinates, the singularities at $r = 0$ are straight lines that stretch from time-like infinity in one asymptotic region to time-like infinity in the other. Now we can draw the Penrose Diagram for maximally extended Schwarzschild solution obtained through the conformal transformations shown above.

Figure 6 (d)

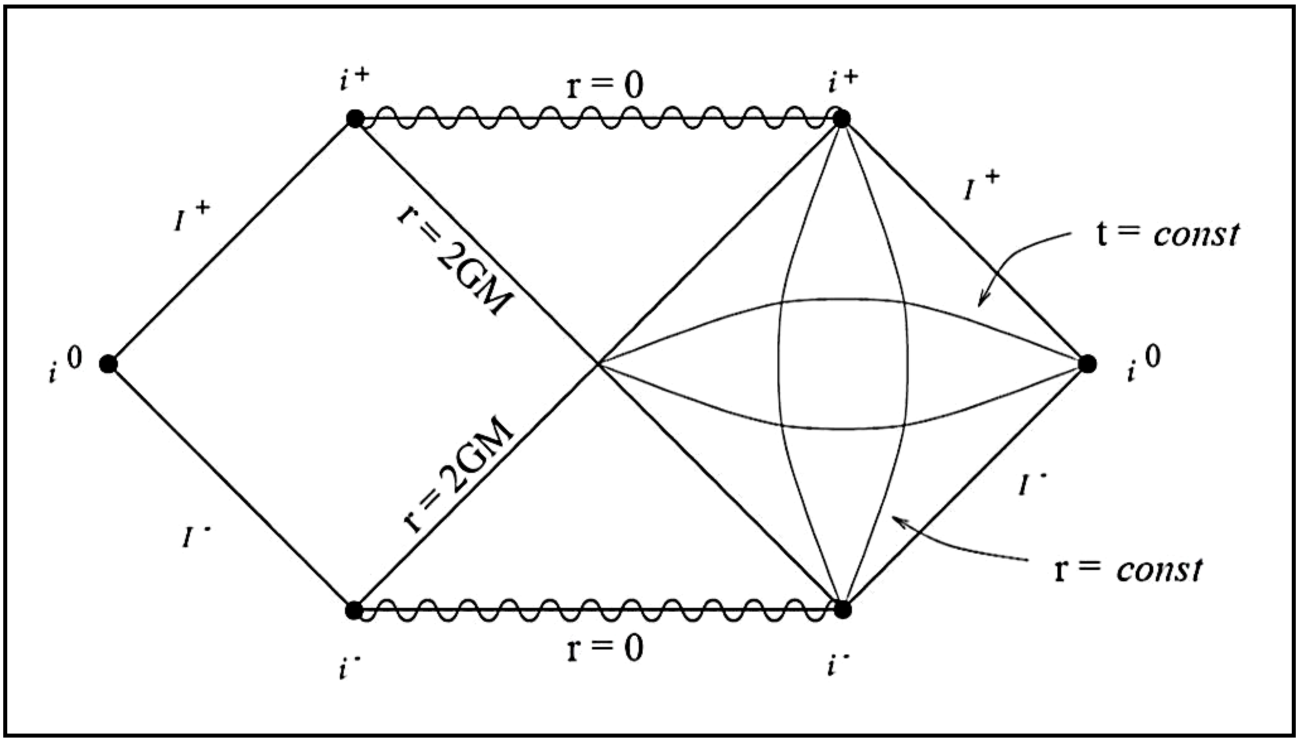


Figure 6(d) shows the Penrose diagram of the Schwarzschild metric. In the figure, $\boldsymbol{i^+}$ is known as future the time-like infinity and $\boldsymbol{i^-}$ is known as past timelike infinity. The time-like geodesic starts from $\boldsymbol{i^-}$ and terminates at $\boldsymbol{i^+}$. $\boldsymbol{i^0}$ is called the space-like infinity. It should be noted that $\boldsymbol{i^+}, \boldsymbol{i^-}$ are distinct points from r = 0. $\boldsymbol{I^+}$ Is called future null infinity and $\boldsymbol{I^-}$ is called past null infinity. We find that there is an event horizon for every observer in region I. We may call this an eternal Black hole. This feature is absent in the extended diagram of the Minkowski space. There is an event horizon for accelerating observer. But, in this case, event horizon exists for non-accelerating observer too. We shall now show the Kruskal diagram of a collapsing star and discuss its physical implications.

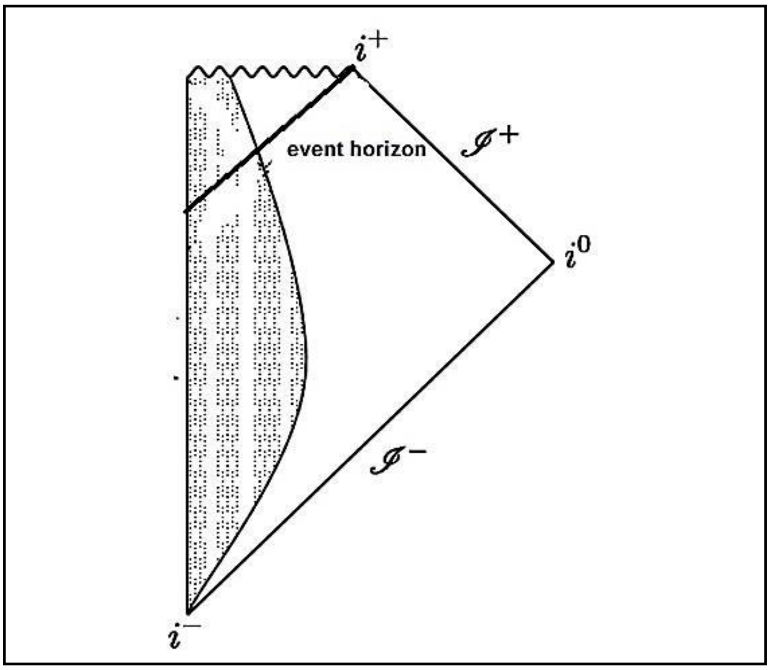


Figure 6 (e)

Figure 6(e) shows the Kruskal diagram of a collapsing star resulting in the formation of a Black Hole. A star can collapse under its own gravitational pull, and shrink down below $\boldsymbol{r = 2M}$ and eventually end up in the formation of singularity- resulting in a blackhole. The shaded region shown in the figure is the interior of the star. The Event horizon is also shown in this figure. It is clear that the event horizon is within the collapsing star. The star will shrink down below the event horizon and then to the singularity. We can consider two observers- one being a distant observer and other being on the surface of the collapsing star. In the figure, the shaded region is the interior of the star. The hyperbola at $r > 2M$ represents the geodesic of the static observer at large distance. Now let us consider that the observer on the collapsing star is sending a signal at regular intervals and the distant observer is receiving it. Then these signals are received by the static observer at longer and longer intervals of proper time as collapse progresses. The last signal to reach the distant observer is when the observer will be at $r = 2M$. Once the observer on the collapsing star crosses the horizon, the signal will be lost forever in the singularity. The distant observer will not receive anything.

Chapter 7

CHARGED BLACK HOLES

The Reissner–Nordström geometry

So far we have roughly discussed all the aspects of neutral, static and spherically symmetric Black Holes along with their various properties and physical implications. We shall now turn our attention to charged Black Holes and reinvestigate the space-time geometry by modifying Einstein's field equations afresh in this new perspective. So we shall now consider the space-time geometry outside a static, spherically symmetric charged object; once again this is not based in vacuum, since it is filled with a static electric field whose energy–momentum must be included in the field equations.

The famous static solution to Einstein's Field equations which describes the geometry of the space-time surrounding a non-rotating charged spherical black hole is called the $\boldsymbol{Reissner - Nordstr\ddot{o}m\ metric}$. We must also keep in mind that in reality a highly charged black hole would be quickly neutralized by interactions with matter in its vicinity and therefore such a solution is not extremely relevant to realistic astrophysical situations. Nevertheless, charged black holes illustrate a number of important features of more general situations and pave a way to our understanding of rotating black holes and their inherent mechanisms. However we shall now focus our attention here on deriving the $\boldsymbol{Reissner - Nordstr\ddot{o}m\ metric}$ assuming existence of the magnetic monopoles along with electrical charge. To this end, we shall need to solve the coupled Einstein-Maxwell equations. Because of the spherical symmetry, the Birkhoff's theorem suggests the following generic form for the metric in 4D spherical coordinates $\{\boldsymbol{t}, \boldsymbol{r}, \boldsymbol{\theta}, \boldsymbol{\varphi}\}$:

$$ds^2 = -e^{2\alpha(r,t)}dt^2 + e^{2\beta(r,t)}dr^2 + r^2 d\Omega^2 \tag{7.1}$$

Where, $d\Omega^2$ is the metric on a unit two dimensional sphere given by:

$$d\Omega^2 = d\theta^2 + sin^2\theta d\varphi^2 \tag{7.2}$$

Now, we already know that Einstein's equation for general relativity is:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{7.3}$$

where $R_{\mu\nu}$ is the Ricci tensor obtained from the Riemann curvature tensor,

$$R^{\alpha}_{\mu\lambda\nu} = \partial_\lambda \Gamma^{\alpha}_{\mu\nu} - \partial_\nu \Gamma^{\alpha}_{\mu\lambda} + \Gamma^{\alpha}_{\lambda\rho}\Gamma^{\rho}_{\mu\nu} - \Gamma^{\alpha}_{\nu\rho}\Gamma^{\rho}_{\lambda\mu,} \tag{7.4}$$

By contracting λ with α. R is the Ricci scalar $R = g_{\mu\nu}R_{\mu\nu}$. However, we must keep in mind that in this case unlike the Schwarzschild metric, $R_{\mu\nu}$ is not equal to zero. The R.H.S. of the Einstein Field equations contain the energy-momentum tensor $T_{\mu\nu}$ which in our problem is the one for electromagnetism and it is written as:

$$T_{\mu\nu} = F_{\mu\rho}F^{\nu}_{\rho} - \frac{1}{4} g_{\mu\nu}F_{\rho\sigma} F^{\rho\sigma} \tag{7.5}$$

Where $F_{\mu\nu}$ is the electromagnetic field strength tensor and $T_{\mu\nu}$ has zero trace, which implies:

$$T = g^{\mu\nu}T_{\mu\nu} = g^{\mu\nu}F_{\mu\rho}F^{\nu}_{\rho} - \frac{1}{4}g^{\mu\nu}g_{\mu\nu}F_{\rho\sigma} \ F^{\rho\sigma} = 0$$

since in 4-dimensions $g^{\mu\nu}g_{\mu\nu} = 4$. In Eq. (7.4) Γ's are the connection coefficients given by

$$\Gamma^{\sigma}{}_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}\left(\partial_{\mu}g_{\nu\rho} + \partial_{\nu}g_{\rho\mu} - \partial_{\rho}g_{\mu\nu}\right), \tag{7.6}$$

The fact that $T_{\mu\nu}$ has zero trace, allows us to rewrite the Einstein's equation in the following form:

$$R_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{7.7}$$

Finally, the Maxwell's equations are: $g^{\mu\nu}\nabla_{\mu}F_{\nu\sigma} = 0$ (7.8)

Where, ∇ is the covariant derivative operator and the covariant derivative of a rank two tensor $T_{\nu\sigma}$ is defined to be: $\nabla_{\mu}T^{\nu\sigma} = \partial_{\mu}T^{\nu\sigma} + \Gamma^{\sigma}_{\mu\lambda}T^{\lambda\nu} + \Gamma^{\nu}_{\mu\lambda}T^{\sigma\lambda}$ (7.9)

Now, we shall analyse the components of the electro-magnetic field tensor. Since there is spherical symmetry, the only non-zero components of the magnetic and electric fields are the radial components which should be independent of θ and φ. Therefore the radial component of the electric field has a form of $E_r = F_{tr} = -F_{rt} = f(r,t)$ (7.10)

Whereas, the radial component of the magnetic field is given by:

$$B_r = g_{11}\epsilon^{01\mu\nu} F_{\mu\nu} = \frac{g_{11}}{\sqrt{|g|}}\epsilon^{-01\mu\nu} F_{\mu\nu}$$

$$\frac{g_{11}}{\sqrt{|g|}}(\epsilon^{0123}F_{23} + \epsilon^{-0132}F_{32}) = 2\frac{g_{11}}{\sqrt{|g|}}F_{\theta\varphi} \tag{7.11}$$

From Eq.(7.1)we see that $g_{11} = g_{rr}(r,t)$ and $|g| \propto r^4 \sin^2\theta$ and since B_r doesn't have angular dependence, $F_{\theta\varphi}$ must have the following form: $F_{\theta\varphi} = -F_{\varphi\theta} = g(r,t)r^2 \sin\theta$. All the remaining components of the electromagnetic field strength tensor are either zero or related to these two through symmetries. Therefore for the electromagnetic field strength tensor we obtain,

$$F_{\mu\nu} = \begin{pmatrix} 0 & f(r,t) & 0 & 0 \\ -f(r,t) & 0 & 0 & 0 \\ 0 & 0 & 0 & g(r,t)r^2\sin\theta \\ 0 & 0 & -g(r,t)r^2\sin\theta & 0 \end{pmatrix} \tag{7.12}$$

Now we shall utilize the Ricci tensor components that we had calculated for the Schwarzschild solution which are:

1. $R_{tt} = [\partial_t^2 \beta + (\partial_t\beta)^2 - \partial_t\alpha\partial_t\beta] + e^{2(\alpha-\beta)}[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\alpha$ (7.13)
2. $R_{rr} = -\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta - \frac{2}{r}\partial_r\beta\right] + e^{2(\beta-\alpha)}[\partial_t^2 \beta + (\partial_t\beta)^2 - \partial_t\alpha\partial_t\beta]$
3. $R_{tr} = \frac{2}{r}\partial_t\beta$

4. $R_{\theta\theta} = e^{-2\beta}[r(\partial_r\beta - \partial_r\alpha) - 1] + 1$
5. $R_{\varphi\varphi} = R_{\theta\theta}\, sin^2$

For the components of the electromagnetic stress tensor using Eqs. (7.5) and (7.12) we obtain:

(7.14)

1. $T_{tt} = \frac{f(r,t)^2}{2} e^{-2\beta(r,t)} + \frac{g(r,t)^2}{2} e^{2\alpha(r,t)}$
2. $T_{rr} = -\frac{f(r,t)^2}{2} e^{-2\alpha(r,t)} - \frac{g(r,t)^2}{2} e^{2\beta(r,t)}$
3. $T_{tr} = 0$
4. $T_{\theta\theta} = \frac{r^2 g(r,t)^2}{2} + \frac{r^2 f(r,t)^2}{2} e^{-2(\alpha(r,t)+\beta(r,t))}$
5. $T_{\varphi\varphi} = T_{\theta\theta}\, sin^2\theta$

From Eqs. (7.13) and (7.14) we have $R_{tr} = 0$ which gives $\beta = \beta(r)$. Using this fact and Eq. (7.10), we obtain $e^{2\alpha(r,t)} R_{rr} + e^{2\beta(r)} R_{tt} = 0$. Solving this yields $\alpha(r,t) + \beta(r) = const.$ but we can redefine the time coordinate in Eq. (7.1) by replacing $dt \rightarrow e^{const} dt$ so that:

$$\alpha(r,t) = \alpha(r) = -\beta(r) \tag{7.15}$$

Now we shall solve the Maxwell equations for the form of the electromagnetic field strength tensor given in Eq. (7.12). For the r component of the Eq. (7.8) we have:

$$\partial_t F_{tr} - \Gamma^{\alpha}_{tt} F_{\alpha r} - \Gamma^{\alpha}_{tr} F_{t\alpha} = 0 \tag{7.16}$$

or, carrying out the summation over α gives $\partial_t F_{tr} - F_{tr}(\Gamma^{r}_{tt} + \Gamma^{r}_{tr}) = 0$ (7.17)

Since the metric is diagonal and β doesn't depend on time, $\Gamma^{t}_{tt} = 0$ and $\Gamma^{r}_{tr} = \partial_t \beta = 0$ and from above equation we have $\partial_t F_{tr} = 0$ implying that the tr component of the electromagnetic field strength tensor is not time dependent, so $Ftr = f(r)$.

To find the explicit form of f, we will make use of the following identity: for given any anti-symmetric rank two tensor, $T_{\mu\nu}$, and diagonal metric the following identity is true:

$$\nabla_{\mu} T_{\mu\nu} = \frac{1}{\sqrt{|g|}} \partial_{\mu}(\sqrt{|g|} T^{\mu\nu}). \tag{7.18}$$

If we take into account Eq. (20), for our metric we have $|g| = r^2 sin\theta$. Now if we use metric compatibility condition to raise the indices of the electromagnetic field strength tensor in Eq. and apply the above identity, we obtain

$$\nabla_{\mu} F_{\mu\nu} = \frac{1}{r^2 sin\theta} \partial_{\mu}(r^2\, sin\theta F_{\mu\nu}) = 0. \tag{7.19}$$

For the t component we have $\partial_r(r^2 F_{rt}) = \partial_r(r^2 g^{rr} g^{tt} F_{rt}) = \partial_r(r^2 f) = 0$,

$$f(r) = \frac{const}{r^2} \tag{7.20}$$

For the constant the Gauss's flux theorem gives $const. = Q/\sqrt{4\pi}$ where Q is the total electric charge of a black hole. Now we shall find $g(r,t)$ which is related to the magnetic filed. For this we need to solve Eq. (7.8) which in explicit form reads

(7.21)

$$\nabla_\mu F_{\nu\rho} + \nabla_\nu F_{\rho\mu} + \nabla_\rho F_{\mu\nu} = 0$$

If we expand above equation using Eq. (7.9), all terms which contain connection coefficients vanish and we are left with the ordinary partial derivatives

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0. \tag{7.22}$$

Considering $\mu = t, \nu = \varphi$ and $\rho = \theta$ combination and using the fact that $F_{\theta t} = F_{t\varphi} = 0$, we obtain $\partial_t F_{\theta\varphi} = 0$ which means that $g(r,t)$ is time independent. Doing the same for $\mu = r, \nu = \theta$ and $\rho = \varphi$ combination leads to $\partial_r F_{\theta\varphi} = 0$ or $\partial r(r2g(r)) = 0$. Thus

$$g(r,t) =. \frac{const}{r^2} \tag{7.23}$$

Similar to the electric charges, the Gauss's flux theorem for the magnetic field gives $const. = P/\sqrt{4\pi}$ where P the total magnetic charge of black hole is. Finally, for the electromagnetic field strength tensor we obtain

$$F_{\mu\nu} = \frac{1}{\sqrt{4\pi}} \begin{pmatrix} 0 & Qr^{-2} & 0 & 0 \\ -Qr^{-2} & 0 & 0 & 0 \\ 0 & 0 & 0 & P\sin\theta \\ 0 & 0 & -P\sin\theta & 0 \end{pmatrix} \tag{7.24}$$

Now we are left with only one unknown variable, $\alpha(r)$ which is given in equation (7.15) . To this end, one equation is enough to determine the unknown. Let's now consider the $\theta\theta$ component of the Einstein's equation, Eq. (7.6):

$$R_{\theta\theta} = 8\pi T_{\theta\theta} \tag{7.25}$$

Substituting $R_{\theta\theta}$ and $T_{\theta\theta}$ from relations (7.13) and (7.14) into the above equation and using relations (7.15) and (7.24), we obtain:

$$\partial_r(re^{2\alpha}) = 1 - \frac{G}{r^2}(Q^2 + P^2) \tag{7.26}$$

Or,

$$e^{2\alpha} = 1 + \frac{R_S}{r} + \frac{G}{r^2}(Q^2 + P^2) \tag{7.27}$$

In the absence of charges, this should reduce to the Schwarzschild solution which allows us to take the constant to be $R_S = 2GM$ where **M** is interpreted as the mass of black hole and **G** is the Newton's gravitational constant. Finally, upon substitution of equations (7.15) and (7.27) into Equation (7.1), the ***Reissner − Nordström Metric*** is readily found:

$$ds^2 = -\Delta dt^2 + \Delta^{-1} dr^2 + r^2 d\Omega^2 \tag{7.28}$$

Where;

$$\Delta = 1 - \frac{2GM}{r} + \frac{G}{r^2}(Q^2 + P^2) \tag{7.29}$$

In summary, we have solved the coupled Einstein-Maxwell equations and found the metric which describes the geometry of the space-time surrounding a nonrotating black hole assuming it has static electric and magnetic charges.

Properties of Charged Black Holes in RN geometry

The Reissner–Nordström (RN) metric shown in relations (7.28) and (7.29) is only valid down to the surface of the charged object. As in our discussion of the Schwarzschild solution, however, it is of interest to consider the structure of the full RN geometry, namely the solution to the coupled Einstein–Maxwell field equations for a *charged point mass* located at the origin $r = 0$, in which case the RN metric is valid for all positive r. Calculation of the invariant curvature scalar $\boldsymbol{R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}}$ shows that the only intrinsic singularity in the RN metric occurs at $r = 0$. In the 'Schwarzschild-like' coordinates$(\boldsymbol{t}, \boldsymbol{r}, \boldsymbol{\theta}, \boldsymbol{\varphi})$, however, the RN metric also possesses a coordinate singularity wherever r satisfies

$$\Delta(\boldsymbol{r}) = \boldsymbol{1} - \frac{\boldsymbol{2GM}}{\boldsymbol{r}} + \frac{\boldsymbol{G}}{\boldsymbol{r^2}}(\boldsymbol{Q^2} + \boldsymbol{P^2}) = \boldsymbol{0} \tag{7.30}$$

with $\Delta(\boldsymbol{r}) = -\boldsymbol{1}/\boldsymbol{g_{11}}(\boldsymbol{r}) = \boldsymbol{g_{00}}(\boldsymbol{r})/\boldsymbol{c2}.$ Multiplying equation (7.30) throughout by r^2 and solving the resulting quadratic equation, we find that the coordinate singularities occur on the surfaces $r = r\pm$, where:

$$\boldsymbol{r}\pm= \boldsymbol{M} \pm (\boldsymbol{M^2} - \boldsymbol{q^2})^{\frac{1}{2}} \tag{7.31}$$

The above equation is written in geometrized units. However, it is clear that there exist three distinct cases, depending on the relative values of M^2 and q^2; we now discuss these in turn.

- $\boldsymbol{Case\ 1}$: $\boldsymbol{M^2} < \boldsymbol{q^2}$: In this case $r\pm$ are both imaginary, and so no coordinate singularities exist. The metric is therefore regular for all positive values of r. Since the function r always remains positive, the coordinate t is always time-like and r is always space-like. Thus, the intrinsic singularity at $\boldsymbol{r} = \boldsymbol{0}$ is a time-like line, as opposed to a space-like line in the Schwarzschild case. This means that the singularity does not necessarily lie in the future of time-like trajectories and so, in principle, can be avoided. In the absence of any event horizons, however, $\boldsymbol{r} = \boldsymbol{0}$ is a naked singularity, which is visible to the outside world. The physical consequences of a naked singularity, such as the existence of closed time-like curves, appear so extreme that Penrose has suggested the existence of a cosmic censorship hypothesis, which would only allow singularities that are hidden behind an event horizon. As a result, the case $\boldsymbol{M^2} < \boldsymbol{q^2}$ is not considered physically realistic.

- $\boldsymbol{Case\ 2}$: $\boldsymbol{M^2} > \boldsymbol{q^2}$: In this case, $r\pm$ are both real and so there exist two coordinate singularities, occurring on the surfaces $r = r\pm$. the situation at $r = r+$ is very similar to the Schwarzschild case at $\boldsymbol{r} = \boldsymbol{2M}$. For $r > r+$, the function r is positive and so the coordinates t and r are time-like and space-like respectively. In the region $r- < r < r+$, however, r becomes negative and so the physical natures of the coordinates t and r are interchanged. Thus, a massive particle or photon that enters the surface $r = r+$ from outside must necessarily move in the direction of decreasing r, and thus $r = r+$ is an event horizon. The major difference from the Schwarzschild geometry is that the irreversible in fall of the particle need only continue to the surface $r = r-$, since for $r < r-$ the function r is again positive and so t and r recover their timelike and spacelike properties. Within $r = r-$, one may (with a rocket engine) move in the direction of either positive or negative r, or stand still. Thus, one may avoid the intrinsic singularity at $r = 0$, which is consistent with the fact that $r = 0$ is a time-like line. Perhaps even more astonishing is what happens if one then chooses to travel back in the direction of positive r in the region $r < r\ -$.On performing a maximal analytic extension of the RN geometry, in analogy with the Kruskal extension for the Schwarzschild geometry discussed before, we find that we may re-cross the surface $r\ = r -$, but this time from the inside. Once again one is moving from a region in which r is space-like to a region in which it is time-like, but this time the sense is reversed and one is forced to move in the direction of increasing r. Thus $r\ = r -$ acts as an 'inside-out' event horizon. Moreover, one is eventually ejected from the surface $r\ = r +$ but, according to the maximum analytic extension, the particle emerges into a asymptotically flat spacetime different from that from which it first entered the black hole.

- ***Case* 3**: $\boldsymbol{M^2 = q^2}$: In this case, called the extreme Reissner–Nordström black hole, the function $\Delta(\boldsymbol{r})$ is positive everywhere except at $\boldsymbol{r\ = M}$, where it equals zero. Thus, the coordinate r is everywhere space-like except at $\boldsymbol{r\ = M}$, where it becomes null, and hence $\boldsymbol{r\ = M}$ is an event horizon. The extreme case is basically the same as that considered in case 2, but with the region $\boldsymbol{r- < r < r+}$ removed.

We may illustrate the properties of the RN space-time in more detail by considering the paths of photons and massive particles in the geometry, which we now go on to discuss. Since the case $\boldsymbol{M^2 > q^2}$ is the most physically reasonable RN space-time, we shall restrict our discussion to this situation.

Radial Photon Trajectories in RN Geometry

Let us begin by investigating the paths of radially incoming and outgoing photons in the RN metric for the case $\boldsymbol{M^2 > q^2}$. Since $\boldsymbol{ds = d\theta = d\varphi = 0}$ for a radially moving photon, we have immediately from (7.31) that:

$$\frac{\boldsymbol{dt}}{\boldsymbol{dr}} = \pm\frac{\boldsymbol{1}}{\boldsymbol{c}} = \left(\boldsymbol{1} - \frac{\boldsymbol{2M}}{\boldsymbol{r}} + \frac{\boldsymbol{q^2}}{\boldsymbol{r^2}}\right)^{\boldsymbol{-1}} = \pm\frac{\boldsymbol{1}}{\boldsymbol{c}}\left(\frac{\boldsymbol{r^2}}{(\boldsymbol{r - r_-})(\boldsymbol{r - r_+})}\right) \tag{7.32}$$

where, in the second equality, we have used the result (7.31); the plus sign corresponds to an outgoing photon and the minus sign to an incoming photon. On integrating, we obtain:

$$\boldsymbol{ct = r - \frac{r_-^2}{r_+ - r_-} ln\left|\frac{r}{r_-} - 1\right| + \frac{r_+^2}{r_+ - r_-}\ ln\left|\frac{r}{r_+} - 1\right| + constant} \qquad (\boldsymbol{outgoing}) \tag{7.33}$$

$$\boldsymbol{ct = -r + \frac{r_-^2}{r_+ - r_-} ln\left|\frac{r}{r_-} - 1\right| + \frac{r_+^2}{r_+ - r_-}\ ln\left|\frac{r}{r_+} - 1\right| + constant} \qquad (\boldsymbol{ingoing})$$

We will concentrate in particular on the ingoing radial photons. To develop a better description of in-falling particles in general, we may construct the equivalent of the advanced Eddington–Finkelstein coordinates derived for the Schwarzschild metric. Once again this coordinate system is based on radially in-falling photons, and the trick is to use the integration constant as the new coordinate, which we denote by p. As before, p is a null coordinate and it is more convenient to work instead with the time-like coordinate t' defined by $ct' = p - r$. Thus, we have:

(7.34)

$$ct' = ct - \frac{r_-^2}{r_+ - r_-} ln\left|\frac{r}{r_-} - 1\right| + \frac{r_+^2}{r_+ - r_-} ln\left|\frac{r}{r_+} - 1\right|$$

On differentiating, or from (7.32) directly, we obtains

$$cdt' = dp - dr = cdt + \left[\frac{1}{\Delta(r)} - 1\right] dr \quad (7.35)$$

Where $\Delta(r)$ defined in (7.30). Using the above expression to substitute for c in (7.29), one quickly finds that:

$$ds^2 = c^2\Delta\, dt'^2 - 2(1-\Delta)dt'dr - (2-\Delta)dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \quad (7.36)$$

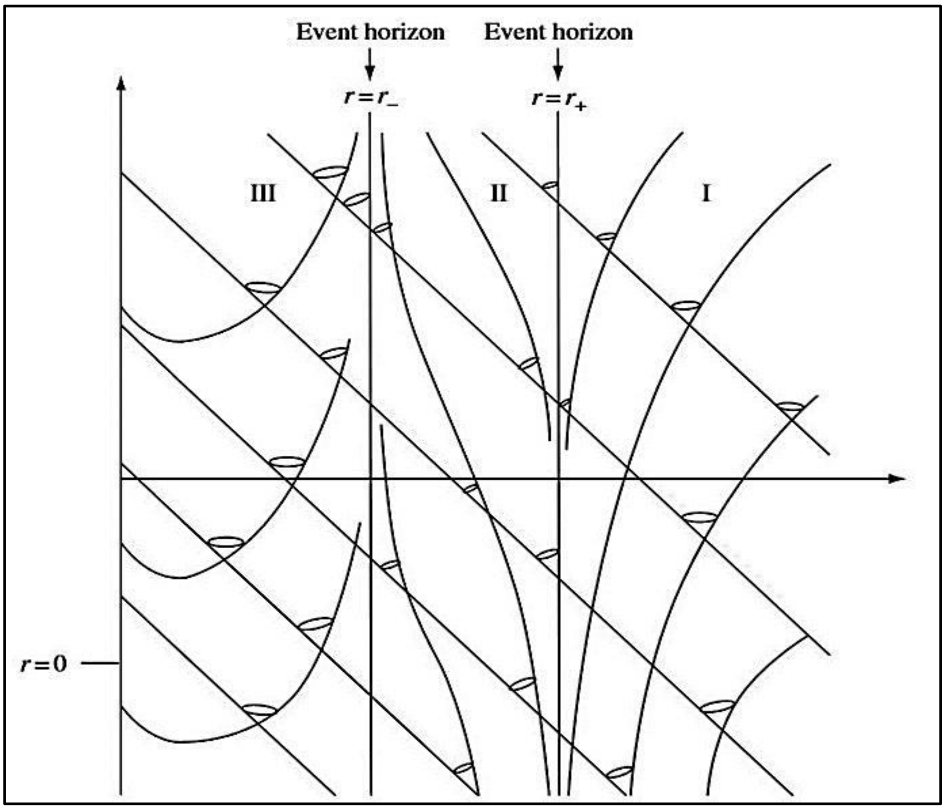
Which is the RN metric in advanced Eddington–Finkelstein coordinates. In particular, we note that this form is regular for all positive values of r and has an intrinsic singularity at $r = 0$. From (7.32) and (7.35), one finds that, in advanced Eddington–Finkelstein coordinates, the equation for ingoing radial photon trajectories is:

$$ct' + r = constant \quad (7.37)$$

Whereas the trajectories for outgoing radial photons satisfy the differential equation:

$$\frac{cdt'}{dr} = \frac{2-\Delta}{\Delta} \quad (7.38)$$

Figure 7 (a)



The above figure shows the space-time diagram of the ***Reissner–Nordström*** solution in advanced Eddington–Finkelstein coordinates. The straight diagonal lines are ingoing photon worldliness whereas the curved lines correspond to outgoing photon worldlines.

We may use these equations to determine the light-cone structure of the RN metric in these coordinates. For ingoing radial photons, the trajectories (7.37) are simply straight lines at 45° in a space-time diagram. For outgoing radial photons, (7.38) gives the gradient of the trajectory at any point in the space-time diagram, and so one may sketch these without solving (7.38) explicitly. This resulting space-time diagram is shown in **Figure 7(a).** It is worth noting that the light-cone structure depicted confirms the nature of the event horizon at $r = r+$. Moreover, the lightcones remain tilted over in the region $r- < r < r+$, indicating that any particle falling into this region must move inwards until it reaches $r = r-$. Once in the region $r < r-$, the lightcones are no longer tilted and so particles need not fall into the singularity $r = 0$.As was the case for the Schwarzschild metric, however, this space-time diagram may be somewhat misleading. For an outwardmoving particle in the region $r < r-$, **Figure 7(a)** suggests that it can only reach $r = r-$ asymptotically, but by peforming an analytic extension of the RN solution one can show that the particle can cross the surface $r = r-$ in finite proper time.

Penrose Diagrams of RN Solution

Just as we had done Penrose Diagrams to depict the Schwarzschild Black hole using Kruskal Szekers coordinates and suitable conformal transformations, similarly following the same analogy we shall construct the Penrose Diagrams for the various regions of the Reissner–Nordström solution as discussed in the previous section. Let us first write the Reissner–Nordström metric that has been previously derived in geometrized units and considering only the presence of electric charge:

$$\boldsymbol{ds^2 = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2} \tag{7.39}$$

Where $\boldsymbol{d\Omega^2 = (d\theta^2 + \sin^2\theta\, d\varphi^2)}$

Now, as we have seen earlier; multiplying the above equation throughout by r^2 and solving the resulting quadratic equation we can find its coordinate and intrinsic singularities. Thus we get: $\boldsymbol{\frac{1}{r^2}(r^2 - 2Mr + q^2) = 0}$ Here $\boldsymbol{q^2 = Q^2}$**.** (In case of the previous relation) So solving this we obtain singularities at $\boldsymbol{r = r\pm}$ such that:

$$\boldsymbol{r = M \pm \sqrt{M^2 - q^2}} \tag{7.40}$$

The number of real roots in this case will depend on the positivity of $\boldsymbol{\sqrt{M^2 - q^2}}$ which will be true when $\boldsymbol{M^2 > q^2}$ and $\boldsymbol{M^2 = q^2}$**.** Now for the time being, we shall only deal with the real solutions keeping aside the imaginary ones in case $\boldsymbol{M^2 < q^2}$**.** So the total number of possibilities in our case will be:

1) Two real roots for $\boldsymbol{M > |q|}$. Such that $\boldsymbol{r\pm = M + \sqrt{M^2 - q^2}}$**.**
2) One double real root for $\boldsymbol{M = |q|}$**.** Such that $\boldsymbol{r = M}$**.**
3) No real roots.

Case 1: Now let us consider the 1st scenario; in this case, the 2 real roots are r_+ and r_-. We have to write the metric in a convenient form such that:

$$ds^2 = \frac{1}{r^2}(r - r_+)(r - r_-)dt^2 + r^2(r - r_+)(r - r_-)^{-1}dr^2 + r^2\, d\Omega^2 \tag{7.41}$$

Due to spherical symmetry we only consider the above portion of the equation for our calculations which is called the Lorentzian part of the metric. So we have:

$$\frac{1}{r^2}(r - r_+)(r - r_-)dt^2 + r^2(r - r_+)(r - r_-)^{-1}dr^2 \tag{7.42}$$

Let us now consider the radial component such that:

$$\frac{r dr}{\sqrt{(r - r_+)(r - r_-)}} = \frac{1}{r}\sqrt{(r - r_+)(r - r_-)} \tag{7.43}$$

Doing this transformation we bring the metric to a conformally flat form so that:

$$\begin{aligned} ds^2 &= \frac{1}{r^2}(r - r_+)(r - r_-)dt^2 + r^2(r - r_+)(r - r_-)^{-1}dr^2 \\ &= \frac{1}{r^2}(r - r_+)(r - r_-)\,(-dt^2 + dR^2) \end{aligned} \tag{7.44}$$

We must keep in mind the distinct coordinate regions that are generated form the roots of the solution; which are: 1) $r > r_+$; 2) $r_- < r < r_+$; 3) $0 < r < r_-$. Now we shall compare and analyse with respect to our previously obtained Penrose diagram for flat space as shown below in Figure 7 (b):

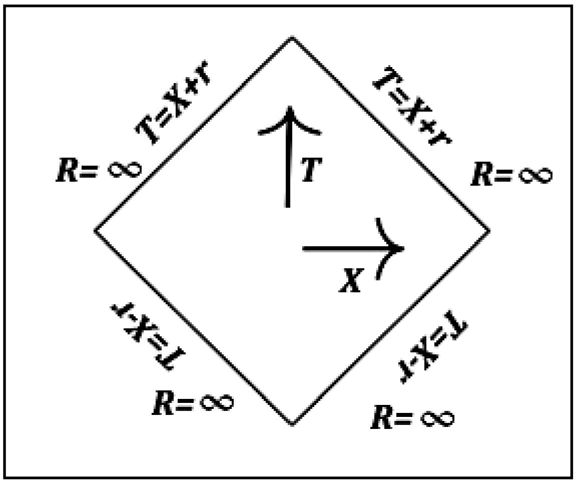


Figure 7 (b)

The above diagram, Figure 7(b) has the following limits:

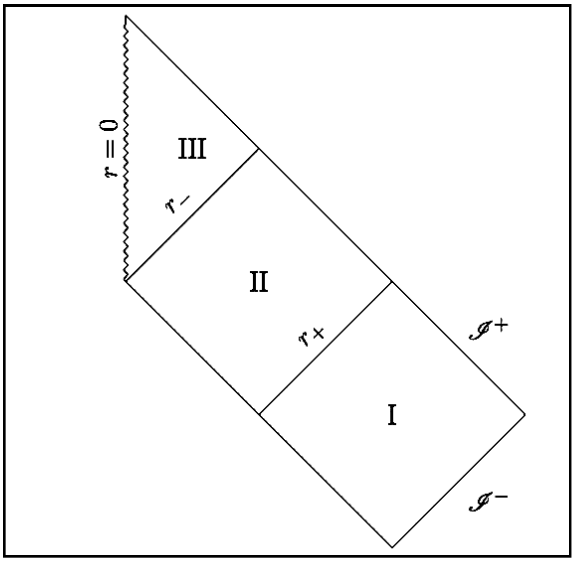
1) $\lim_{T\to X+r} R(T,X) = +\infty \quad \left(\frac{T+X}{2} < \frac{\pi}{2}, X > 0\right)$
2) $\lim_{T\to X-r} R(T,X) = +\infty \quad \left(\frac{T-X}{2} < \frac{\pi}{2}, X > 0\right)$
3) $\lim_{T\to X+r} R(T,X) = -\infty \quad \left(\frac{T-X}{2} < \frac{\pi}{2}, X < 0\right)$
4) $\lim_{T\to X-r} R(T,X) = -\infty \quad \left(\frac{T+X}{2} < \frac{\pi}{2}, X < 0\right)$

Now from the transformation of the radial part we get: $dR = r^2\left(\frac{dr}{(r-r_+)(r-r_-)}\right)$ (7.45)

Integrating this we get: $R = \int \frac{r^2 dr}{(r-r_+)(r-r_-)}$ or $R = r + \frac{1}{(r-r_+)}\left(r_+^2 \ln|r - r_+| - r_-^2 \ln|r - r_-|\right)$ (7.46)

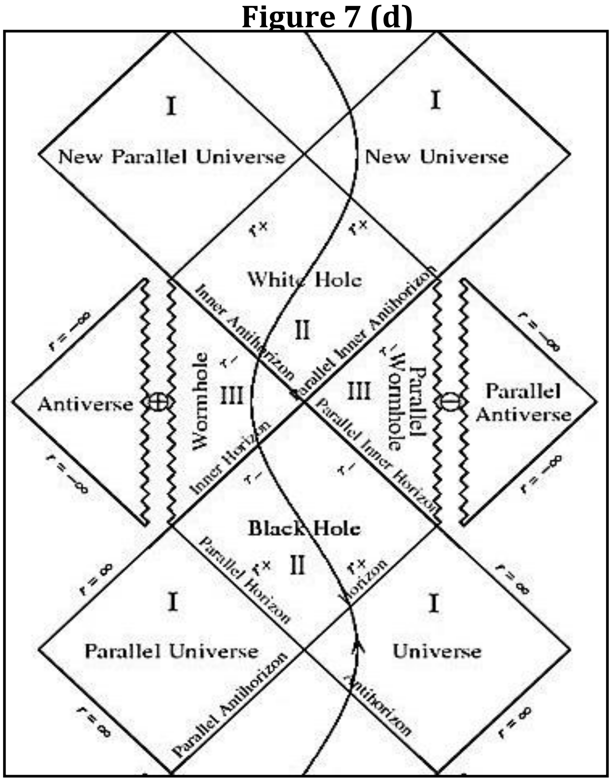
Now we can carefully consider the three regions and find the corresponding Penrose Diagram by comparing the limits with the above diagram. The first region has limits $\boldsymbol{lim_{r\to\infty}R(r)} = \infty$ and $\boldsymbol{lim_{r\to r_+}R(r)} = -\infty$ .The second region has the limits $\boldsymbol{lim_{r\to r_+}R(r)} = -\infty$ And $\boldsymbol{m_{r\to r_-}R(r)} = \infty$ **.** While the third one has $\boldsymbol{lim_{r\to r_-}R(r)} = \infty$ and in case of the other side the diagram terminates at $\boldsymbol{r = 0}$**.** Now that we have obtained an overview of all the three regions, we can construct suitable Penrose Diagrams for each region and combine them to obtain the complete Penrose diagram of the Reissner–Nordström solution for the 1St case such that; $\boldsymbol{M > |q|}$.

Figure 7 (c)



In **Figure 7(c)** $\boldsymbol{I^+ = future\ null\ infinity}$ and $\boldsymbol{I^- = past\ null\ infinity}$. $\boldsymbol{r_+}$ is the **Event Horizon** while $\boldsymbol{r_-}$ is called the **Cauchy Horizon**. $\boldsymbol{r = 0}$ is a real physical singularity; however unlike in case of the Schwarzschild Black-Hole , this singularity is time-like and not space-like hence it can be avoided by an in-falling object. Furthermore the interesting part is that we can create a repetitive sequence of such Penrose Diagrams shown in Figure **7(c)** in the following way as shown next in **Figure 7(d)** which basically represents an infinite sequence of Penrose Diagrams with a some unusual Physical significance.

Figure 7 (d)

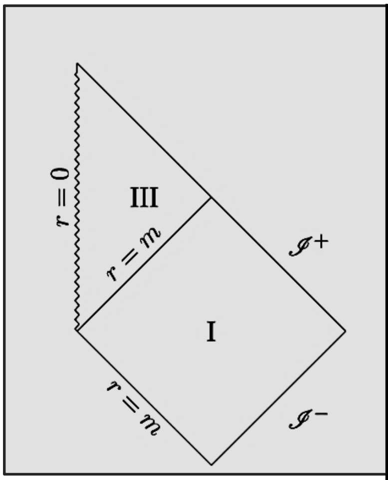


In **Figure 7(d)** all $\boldsymbol{r} = \infty$ represent infinite number of asymptotically flat regions of space. Thereby, giving rise to the notion of multiple and parallel universes. As we have stated earlier, the singularity at $\boldsymbol{r} = \boldsymbol{0}$ is time-like unlike in case of the Schwarzschild solution. Information can even be created at singularity and it can pass through other universes through the Worm Hole. A Worm Hole is essentially a bridge or tunnel like physical solution that connects two different regions of space-time. The second horizon is called the Cauchy Horizon and it is the boundary of globally hyperbolic surfaces. However, particles or objects crossing the Cauchy Horizon may not hit the singularity. An observer starting in region I shall enter the Black Hole through the Event Horizon and reach region II. The observer may continue through the Cauchy Horizon into the region where the singularity is present and may avoid the singularity due to its time-like nature. The observer may continue through the White Hole into another asymptotically flat region of space-time. Thus exiting the Black Hole the observer lands up in another Universe. This behaviour is like a Worm Hole connecting two Universes.

Case 2: In the 2nd scenario we consider one double real root. The physical solution in this case results in something called an Extremal Black Hole; since $\boldsymbol{M^2} = \boldsymbol{q^2}$ is an extremal static case which is between the two distinct real roots solution and the naked singularity generated from the imaginary solution (which has no physical significance due to the principle of cosmic censorship therefore it has not been discussed in this article) . So we see that region II has disappeared since it was defined as the region between the two horizons. Regions III and I are only present in this case. The line element in this case is given by:

$$\boldsymbol{ds^2} = -\frac{(\boldsymbol{r}-\boldsymbol{m})^2}{\boldsymbol{r^2}}\,\boldsymbol{dt^2} + \frac{\boldsymbol{r^2}}{(\boldsymbol{r}-\boldsymbol{m})^2}\boldsymbol{dr^2} + \boldsymbol{r^2 d}\Omega^2 \tag{7.47}$$

And the limits of the two regions in the Penrose Diagram are $\boldsymbol{lim_{r\to\infty}R(r)} = \infty$ and $\boldsymbol{lim_{r\to r_+}R(r)} = -\infty$ for the first region and $\boldsymbol{lim_{r\to r\_}R(r)} = \infty$ and $\boldsymbol{r} = \boldsymbol{0}$ for the second region. Thus the resultant Penrose diagram and its infinite sequence is as follows:



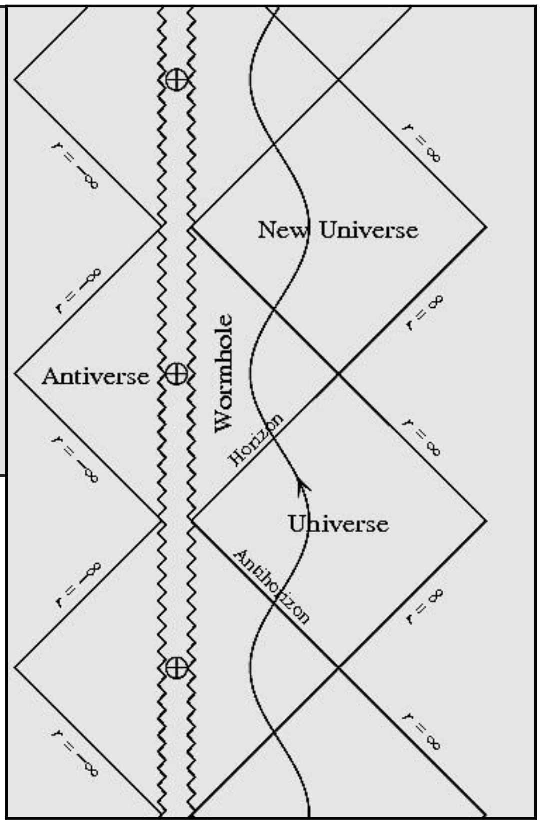


Figure 7 (e)

Chapter 8
ROTATING BLACK HOLES

Investigating the Kerr Solution

The Schwarzschild solution describes the space-time geometry outside a spherically symmetric massive object, characterised only by its mass M. In the previous chapter we derived further spherically symmetric solutions. Most real astrophysical objects, however, are rotating. In this case, a spherically symmetric solution cannot apply because the rotation axis of the object defines a special direction, so destroying the isotropy of the solution .For this reason, in general relativity it is not possible to find a coordinate system that reduces the space-time geometry outside a rotating (uncharged) body to the Schwarzschild geometry. The non-linear field equations couple the source to the exterior geometry. Moreover, a rotating body is characterised not only by its mass **M** but also by its angular momentum **J**, and so we would expect the corresponding space-time metric to depend upon these two parameters.

In this case, in terms of our 'Schwarzschild-like' coordinates $(\boldsymbol{t}, \boldsymbol{r}, \boldsymbol{\theta}, \boldsymbol{\phi})$ the line element for the ***Kerr geometry*** takes the form:

$$\boldsymbol{ds^2 = c^2\left(1 - \frac{2Mr}{\rho^2}\right) dt^2 + \frac{4Macr\,sin^2\theta}{\rho^2}\,dt\,d\phi - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta} \qquad \boldsymbol{- \left(r^2 + a^2 + \frac{2Mra^2\,sin^2\theta}{\rho^2}\right) sin^2\theta d\phi^2} \tag{8.1}$$

Where, $\boldsymbol{M}$ and $\boldsymbol{a}$ are constants and we have introduced the functions Δ and $\boldsymbol{\rho}$, defined by:

$\boldsymbol{\rho^2 = r^2 + a^2cos^2\theta}$ and $\Delta = \boldsymbol{r^2 - 2Mr + a^2}$

This standard expression for $\boldsymbol{ds^2}$ is known as the Boyer–Lindquist form and $(\boldsymbol{t}, \boldsymbol{r}, \boldsymbol{\theta}, \boldsymbol{\phi})$ are known as Boyer–Lindquist coordinates. We can rewrite this above metric in several other useful forms which are more suggestive of a rotational object. We first define 2 functions:

$\Sigma^2 = \boldsymbol{(r^2 + a^2)^2 - a^2}\Delta\boldsymbol{sin^2\theta}$ And $\boldsymbol{\omega = 2Mcra}/\Sigma^2$ **.** Thus we rewrite relation (8.1) in a more convenient form as:

$$\boldsymbol{ds^2 = \frac{\rho^2\Delta}{\Sigma^2}c^2\,dt^2 - \frac{\Sigma^2 sin^2\theta}{\rho^2}(d\phi - \omega dt)^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2} \tag{8.2}$$

The limits of the Kerr Metric depend on the parameters $\boldsymbol{M}$ and $\boldsymbol{a}$ as we might expect for a rotating body. Moreover, in the limit $\boldsymbol{a} \to \boldsymbol{0}$,

$\boldsymbol{\rho^2 \to r^2}$ And $\Sigma^2 \to \boldsymbol{r^4}$.So any of the forms for the Kerr metric above tends to the Schwarzschild form.

$$ds^2 \to c^2\left(1-\frac{2M}{r}\right)dt^2 - \left(1-\frac{2M}{r}\right)dr^2 - r^2\,d\theta^2 - r^2 sin^2\theta d\phi^2 \qquad (8.3)$$

So M in both the cases corresponds to the mass and a corresponds to the angular velocity of the body. The fact that the Kerr metric tends to the Schwarzschild metric as $a \to 0$ allows us to give some geometrical meaning to the coordinates r and θ in the limit of a slowly rotating body.

Dragging of Inertial Frames

Due to the presence of ω in the explicit form of the Kerr metric, it clearly indicates that the source of the gravitational field is rotating. So we have the remarkable result that a particle dropped 'straight in' from infinity $\rho_\phi = 0$ is 'dragged' just by the influence of gravity so that it acquires an angular velocity in the same sense as that of the source of the metric. This effect weakens with distance (roughly as $\sim 1/r^3$ for the Kerr metric) and makes the angular momentum of the source measurable in practice. The effect is called the **dragging of inertial frames.** It is important for us to remember at this point that inertial frames are defined as those in which free-falling test bodies are stationary or move along straight lines at constant speed. Now, let us consider the freely falling particle discussed above. At any spatial point (r, θ, ϕ), in order for the particle to be at rest in some inertial frame the frame must be moving with an angular speed $\omega(r, \theta)$. Any other inertial frame is then related to this instantaneous rest frame by a Lorentz transformation. Thus the inertial frames are 'dragged' by the rotating source.

0

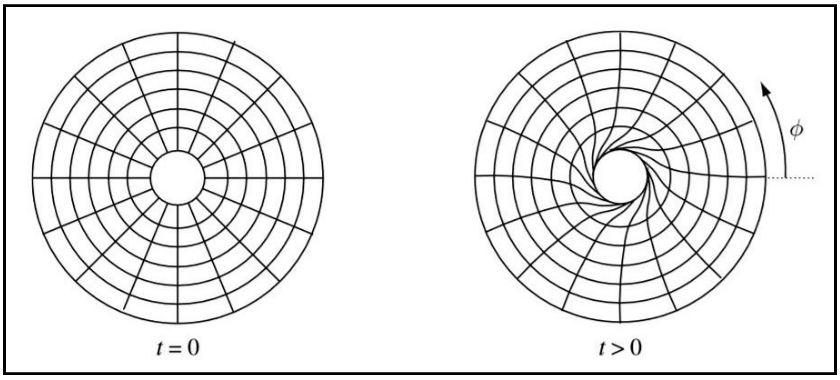


Figure 8 (a)

A schematic illustration of this effect in a plane where $\theta = constant$ is shown in Figure 8(a), where the spacetime around the source is viewed along the rotation axis.

Structure of a Kerr Black Hole

The Kerr metric is the solution to the empty-space field equations outside a rotating massive object and so is only valid down to the surface of the object. As in our discussion of the Schwarzschild solution, however, it is of interest to consider the structure of the full Kerr geometry as a vacuum solution to the field equations.

***Singularities and Horizons*:**

The Kerr metric in Boyer–Lindquist coordinates is singular when $\rho = 0$ and when $\Delta = 0$. Calculation of the invariant curvature scalar $R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}$ reveals that only $\rho = 0$ is an intrinsic singularity. Since:

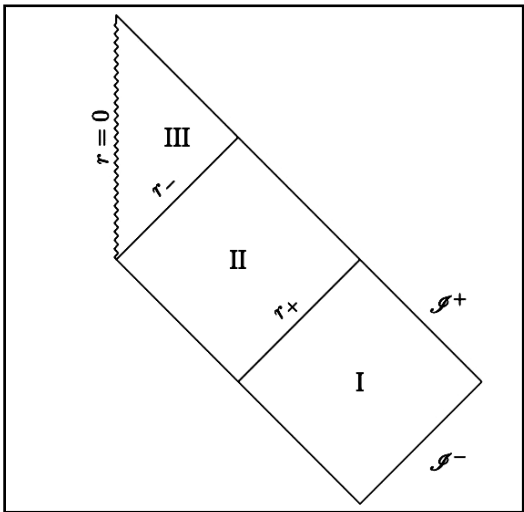
$\rho^2 = r^2 + a^2 cos^2\theta = 0$ .It follows that this occurs when: $r = 0$ *and* $\theta = \frac{\pi}{2}$. From our earlier discussion of Boyer–Lindquist coordinates, we recall that $r = 0$ represents a disc of coordinate radius a in the equatorial plane. Moreover, the collection of points with $r = 0$ and $\theta = \frac{\pi}{2}$ constitutes the outer edge of this disc. Thus, rather surprisingly, the singularity has the form of a ***ring***, of coordinate radius a, lying in the equatorial plane. We also see that in terms of 'Cartesian' coordinates, the singularity occurs when $x^2 + y^2 = a^2$ and $z = 0$. The points where $\Delta = 0$ are coordinate singularities which occur on surfaces:

$$r_\pm = \mu \pm (\mu^2 - a^2)^{\frac{1}{2}} \; where \; \mu = \frac{GM}{c^2} \tag{8.4}$$

Event Horizons in the Kerr metric will occur where $r = constant$ is a null 3-surface, and this is given by the condition $g^{rr} = 0$ or, equivalently, $g_{rr} = \infty$. From the metric itself it is clear that $g_{rr} = -\frac{\rho^2}{\Delta}$; from which we see that the surfaces $r = r+$ and $r = r-$, for which $\Delta = 0$, are in fact event horizons. Thus, the Kerr metric has two event horizons. In the Schwarzschild limit $a \to 0$, these reduce to $r = 2M$ *and* $r = 0$. **The surfaces** $r = r_\pm$ **are axially symmetric, but their intrinsic geometries are not spherically symmetric.** Setting $r = r\pm$ and $t = constant$ in the Kerr metric and noting from (8.4) that $r_\pm^2 + a^2 = 2\mu r_\pm$, we obtain two-dimensional surfaces with the line elements:

$$d\sigma^2 = \rho_\pm^2 d\theta^2 + \left(\frac{2\mu r_\pm}{\rho_\pm}\right)^2 sin^2\theta \, d\phi^2 \tag{8.5}$$

Which do not describe the geometry of a sphere. If one embeds a 2-surface with geometry given by (8.5) in three-dimensional Euclidean space, one obtains a surface resembling an axisymmetric ellipsoid, flattened along the rotation axis. The existence of the outer horizon $r = r+$, in particular, shows that the Kerr geometry represents a **rotating black hole**. It is a one-way surface, like $r = 2\mu$ in the Schwarzschild geometry. Particles and photons can cross it once, from the outside, but not in the opposite direction. It is common practice to define three distinct regions of a Kerr black hole, bounded by the event horizons, in which the solution is regular: Region I, $r+ < r < \infty$; Region II, $r- < r < r+$; and Region III, $0 < r < r-$. Just like constructing Penrose diagrams for the Reissner–Nordström solution, we can construct a similar set of diagrams for the corresponding three regions of a rotating black hole as follows:



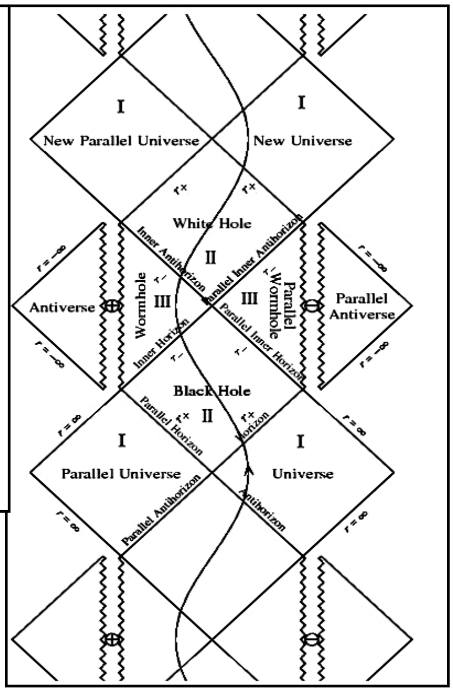


Figure 8 (b)

Not all values of $\boldsymbol{\mu}$ and $\boldsymbol{a}$ correspond to a black hole, however. From (8.5), we see that horizons (at real values of r) exist only for: $\boldsymbol{a^2} < \boldsymbol{\mu}^2$

Thus the magnitude of the angular momentum $\boldsymbol{J} = \boldsymbol{Mac}$ of a rotating black hole is limited by its squared mass. Moreover, as depicted by the various regions in figure 8 (b) if the condition $\boldsymbol{a^2} < \boldsymbol{\mu}^2$ is satisfied then the intrinsic singularity at $\boldsymbol{\rho} = \mathbf{0}$ is contained safely within the outer horizon $\boldsymbol{r} = \boldsymbol{r}+$.

An extreme Kerr black hole is one that has the limiting value $\boldsymbol{a^2} = \boldsymbol{\mu}^2$. In this case, the event horizons $r+$ and $r-$ coincide at $\boldsymbol{r} = \boldsymbol{\mu}$. It may be true that near-extreme Kerr black holes develop naturally in many astrophysical situations. Moreover, matter falling towards a rotating black hole forms an accretion disc that rotates in the same sense as the hole. As matter from the disc spirals inwards and falls into the black hole, it carries angular momentum with it and hence increases the angular momentum of the hole. The process is limited by the fact that radiation from the in-falling matter carries away angular momentum.

For the $\boldsymbol{a^2} > \boldsymbol{\mu}^2$ case, we find that **Δ**>0 throughout, and so the Kerr metric is regular everywhere except at $= \mathbf{0}$, where there is a **ring singularity**. Since the horizons have disappeared, this means that the ring singularity is visible to the outside world. In fact, one can show explicitly that time-like and null geodesics in the equatorial plane can start at the singularity and reach infinity, thereby making the singularity visible to the outside world. Such a singularity is called a naked singularity (as mentioned in the previous discussion about charged Black Holes in Chapter 7) and opens up an enormous realm for some truly wild speculation. However, Penrose's cosmic censorship hypothesis only allows singularities that are hidden behind an event horizon so this case has no real physical significance.

$\boldsymbol{Stationary\ limit\ surfaces\ and\ the\ Ergosphere}$:

In a general stationary axisymmetric space-time the condition $\boldsymbol{g_{tt}} = \mathbf{0}$ defines a surface that is both a stationary limit surface and a surface of infinite redshift. For the Kerr metric, we have:

$$\boldsymbol{g_{tt}} = \boldsymbol{c^2}\left(\mathbf{1} - \frac{\mathbf{2}\boldsymbol{\mu r}}{\boldsymbol{\rho^2}}\right) = \frac{\boldsymbol{c^2(r^2 - 2\mu r + a^2 cos^2\theta)}}{\boldsymbol{\rho^2}} \tag{8.6}$$

so that (for $\boldsymbol{a^2} \leq \boldsymbol{\mu^2}$) these surfaces, S^+ and S^-, occur at:

$$\boldsymbol{r_{S^\pm}} = \boldsymbol{\mu} \pm (\boldsymbol{\mu^2 - a^2 cos^2\theta})^{\frac{1}{2}} \tag{8.7}$$

The two surfaces are axisymmetric, but setting $\boldsymbol{r} = \boldsymbol{r_{S^\pm}}$and $t = constant$ in the Kerr metric, and noting from (8.5) that $\boldsymbol{r^2_{S^\pm}} + \boldsymbol{a^2} = \boldsymbol{2\mu r_{S^\pm}} + \boldsymbol{a^2 sin^2\theta}$, we obtain two-dimensional surfaces with line elements:

$$\boldsymbol{d\sigma^2} = \boldsymbol{\rho_{S^\pm}\, d\theta^2} + \left[\frac{\boldsymbol{2\mu r_{S^\pm}(2\mu r_{S^\pm} + 2a^2 sin^2\theta)}}{\boldsymbol{\rho^2_{S^\pm}}}\right]\boldsymbol{sin^2\theta\, d\phi^2} \tag{8.8}$$

It is clear from the above equation that it does describe the geometry of a sphere. If one embeds a 2-surface with geometry given by (8.8) in three-dimensional Euclidean space then a surface resembling an axisymmetric ellipsoid, flattened along the rotation axis, is once more obtained. In the Schwarzschild limit $a \to 0$, the surface S_+ reduces to $\boldsymbol{r} = \boldsymbol{2\mu}$ and $S\_\ to\ \boldsymbol{r} = \mathbf{0}$. As anticipated we see that, in the Schwarzschild solution, the surfaces of infinite redshift and the event horizons coincide. The surface S_- coincides with the ring singularity in the equatorial plane. Moreover, S_- lies completely within the inner horizon $\boldsymbol{r} = \boldsymbol{r}_-$ (except at the poles, where they touch). The surface S_+ has coordinate radius $\boldsymbol{2\mu}$ at the equator and for all $\boldsymbol{\theta}$ it completely encloses the outer horizon $\boldsymbol{r} = \boldsymbol{r}_+$ (except at the poles, where they touch), giving rise to a region between the two called the **Ergoregion**. The external surface of this region is called the **Ergosphere.**

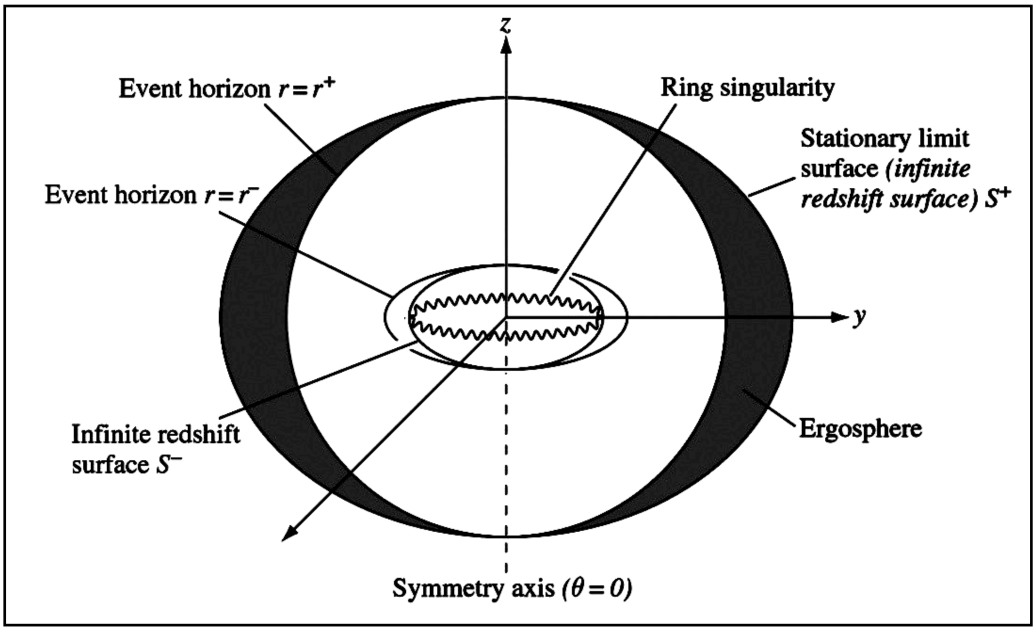


Figure 8 (c)

The Ergoregion gets its name from the Greek word ergo meaning work. The key property of an ergoregion (which can occur in other space-time geometries) is that it is a region for which $\boldsymbol{g_{tt}} < \boldsymbol{0}$ and from which particles can escape. Clearly, the Schwarzschild geometry does not possess an ergoregion, since $\boldsymbol{g_{tt}} < \boldsymbol{0}$ is only satisfied within its event horizon. As we will discuss in the next section, Roger Penrose has shown that it is possible to extract the rotational energy of a Kerr black hole from within the Ergoregion.

The Wormhole Behaviour of a Rotating Black Hole

The general notion of a Wormhole is that it is a structural feature of space-time that behaves like tunnel with two ends, each in separate points in space-time. In other words it is essentially a shortcut linking two asymptotically flat regions of space-time. Now in case of a rotating Black hole which has been illustrated and explained throughout our discussions regarding the Kerr metric; however, so far we have mostly stuck to discussions outside the ring singularity structure of Kerr Black Holes. We have also noticed that this ring singularity is time-like and not space-like unlike the normal Penrose diagram of a Schwarzschild Black Hole.

This feature along with the infinite series of Penrose diagrams that can be generated from the Kerr solution has interesting hypothetical implications. It may be shown that if a particle passes through the interior of the ring singularity then it emerges into another asymptotically flat space-time, but not a copy of the original one. The new space-time is described by the Kerr metric with $\boldsymbol{r} < \boldsymbol{0}$ and hence $\boldsymbol{\Delta}$ never vanishes, so there are no event horizons.

In the new space-time, the region in the vicinity of the ring singularity has the very strange property that it allows the existence of closed time-like curves. For example, consider a trajectory in the equatorial plane that winds around in $\boldsymbol{\phi}$ while keeping $\boldsymbol{t}$ and $\boldsymbol{r}$ constant. The line element along such a path is:

$$ds^2 = -\left(r^2 + a^2 + \frac{2\mu a^2}{r}\right) d\phi^2 \tag{8.9}$$

The above relation is positive if $\boldsymbol{r}$ is negative and small. These are then closed time-like curves, which violate causality and would seem highly unphysical but nonetheless they seem to be an extended feature of the Kerr structure. *If they represent worldlines of observers, then these observers would travel back and meet themselves in the past!* This is simply however an analytical extension and it may seem highly improbable that in practice the gravitational collapse of a real rotating object would lead to such a strange space-time or would be stable enough for this feature to exist.

Penrose Process of Energy Extraction

As mentioned earlier during our description of the Ergoregion, we shall now discuss the Penrose process, by which energy may be extracted from the rotation of a Kerr black hole (or, indeed, from any space-time possessing an ergoregion) in detail. Suppose that an observer, with a fixed position at infinity, for simplicity, fires a particle A into the ergoregion of a Kerr black hole. The energy of particle A, as measured by the observer at the emission event(Ɛ) , is given by:

$$E^{(A)} = p^{(A)}(Ɛ) \cdot u_{obs} = p_t^{(A)}(Ɛ) \tag{8.10}$$

Where $\boldsymbol{p}^{(A)}(Ɛ)$ is the 4-momentum of the particle at this event and $\boldsymbol{u}_{obs}$ is the 4-velocity of the observer, which has components: $[u^{\mu}_{obs}] = (1, 0, 0, 0)$.Suppose now that, at some point in the ergoregion, particle A decays into two particles B and C. By the conservation of momentum, at the decay event (Ɗ) one has:

$$p^{(A)}(Ɗ) = p^{(B)}(Ɗ) + p^{(C)}(Ɗ) \tag{8.11}$$

If the decay occurs in such a way that particle C (say) eventually reaches infinity, a stationary observer there would measure the particle's energy at the reception event (Ʀ) to be:

$$E^{(C)} = p_t^{(C)}(Ʀ) = p_t^{(C)}(Ɗ) \tag{8.12}$$

Where, in the second equality, we have made use of the fact that the covariant time component of a particle's 4-momentum is conserved along geodesics in the Kerr geometry, since the metric is stationary. Similarly, for the original particle we have:

$$p_t^{(A)}(Ɗ) = p_t^{(A)}(Ɛ) \tag{8.13}$$

Thus, the time component of the momentum conservation condition (8.11) may be written as:

$$E^{(C)} = E^{(A)} - p_t^{(B)}(Ɗ) \tag{8.14}$$

Where $\boldsymbol{p_t^{(B)}}$ is also conserved along the geodesic followed by particle B. The key step is now to note that $\boldsymbol{p_t^{(B)}} = \boldsymbol{e_t}.\boldsymbol{p^{(B)}}$, where $\boldsymbol{e_t}$ is the t-coordinate basis vector, whose squared 'length' is given by; $\boldsymbol{e_t}.\boldsymbol{e_t} = \boldsymbol{g_{tt}}$. So if particle B were ever to escape beyond the outer surface of the ergoregion, which is where $\boldsymbol{g_{tt}} > \mathbf{0}$ then $\boldsymbol{e_t}$ would be time-like. Thus, $\boldsymbol{p_t^{(B)}}$ would be proportional to the particle energy as measured by an observer with 4-velocity along the $\boldsymbol{e_t}$ −direction. In this case $\boldsymbol{p_t^{(B)}}$ must therefore be positive, and so (8.14) shows us that $\boldsymbol{E^{(C)}} < \boldsymbol{E^{(A)}}$, therefore, we get less energy out than we put in. However, if the particle B were never to escape the ergoregion but instead fall into the black hole, then it would remain in a region where $\boldsymbol{g_{tt}} < \mathbf{0}$ and so $\boldsymbol{e_t}$ would be space-like instead of being time-like. In this case $\boldsymbol{p_t^{(B)}}$ would be a component of spatial momentum, which might be positive or negative. For decays where it is negative, from (8.14) we see that $\boldsymbol{E^{(C)}} > \boldsymbol{E^{(A)}}$ and so we have extracted energy from the Black Hole. This is the ***Penrose Process of Energy Extraction***.

What are the consequences of the Penrose process for the black hole?

Well for an in falling particle once it has fallen inside the event horizon, it changes both the mass and angular momentum of the Black Hole. For the in-falling particle the Penrose process reduces both the mass as well as the angular momentum of the Black Hole. This is what is meant by saying that the Penrose process extracts rotational energy from the black hole.

Let us now illustrate this with a famous thought experiment designed by Sean Carroll. He says that the idea is simple; if starting from outside the Ergosphere, we arm ourselves with a large rock and leap toward the black hole. If we call the four-momentum of the (person + rock) system $\boldsymbol{p^{(0)\mu}}$, then the energy $\boldsymbol{E^{(0)}}$ is certainly positive and conserved as we move along our geodesic. Once we enter the Ergosphere, we can hurl the rock with all our might, in a very specific way. If we call our momentum $\boldsymbol{p^{(1)\mu}}$ and that of the rock $\boldsymbol{p^{(2)\mu}}$, then at the instant we throw it we have conservation of momentum just as in special relativity:

$$\boldsymbol{p^{(0)\mu}} = \boldsymbol{p^{(1)\mu}} + \boldsymbol{p^{(2)\mu}} \tag{8.15}$$

This basically implies that; $\boldsymbol{E^{(0)}} = \boldsymbol{E^{(1)}} + \boldsymbol{E^{(2)}}$. But, if we imagine that we are arbitrarily strong (and accurate), we can arrange our throw such that $\boldsymbol{E^{(2)}} < \mathbf{0}$. Furthermore, Penrose was able to show that we can arrange the initial trajectory and the throw such that afterwards we follow a geodesic trajectory back outside the horizon into the external universe. Since our energy is conserved along the way, at the end we will have: $\boldsymbol{E^{(1)}} > \boldsymbol{E^{(0)}}$. Thus we will have emerged with more energy than we entered with. This thought experiment along with **figure 8(d)** below justifies Penrose Process of Energy Extraction from the Ergosphere of a rotating Black Hole.

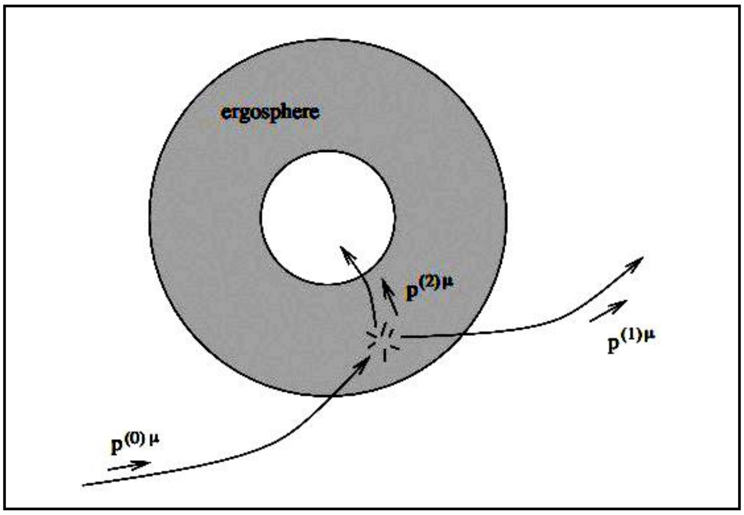


Figure 8 (d)

However we must always keep in mind that energy cannot be created out of nothing. The energy gained has to come from somewhere, and that somewhere is the black hole. In fact, the Penrose process extracts energy from the rotating black hole by decreasing its angular momentum; so we have to throw the rock against the Hole's rotation to get the trick to work.

Conclusion

Hence we have reached the end of our spectacular journey through space-time, decoding the mysterious aspects of one of the greatest enigmas in modern science- The Black Hole. We started with the simple description of flat Minkowski space-time as depicted in Special theory of Relativity and then gradually developed that concept to introduce the various features of General relativity, focused on the principle of Equivalence and the fact that Gravity creates the Geometry of 4-dimensional space-time.

This was followed by an elegant description of curved space-time through the language of tensors which paved a way to introduce the notion of Geodesics. The derivation of the Geodesic equation along with the introduction of Christoffel symbols lead us to the concept of curvature described by the Riemann curvature Tensor and the Ricci tensor. In order to lay the foundation to explore the deeper aspects of this project we have derived Einstein's Field Equations from the Principle of Action using the Curvature Tensor and discussed the physical significance of each term appearing in our final relation.

After completing the background development we shifted our focus to the primary portion of this project which involves all the necessary structural and physical details of the various types of Black Holes and their implications in nature. We started off with deriving the Schwarzschild solution which describes static neutral Black Holes emerging out of Einstein's vacuum Field Equations both using Schwarzschild and Eddington-Finkelstein coordinates. Looking into the concept of Spherical Gravitational Collapse we have compared the idealized solution with the probable realistic scenarios in case of Non-Spherical Gravitational Collapse.

Furthermore we developed the Kruskal extension of the Schwarzschild Metric, thereby modifying our solution in terms of tortoise coordinates in order to facilitate a geometric and diagrammatic representation of the solutions embedded in infinite asymptotically flat space times. Such diagrams which help representing space-time along with all its infinites in a finite 2-dimensional structure are called Penrose Diagrams. We have repeatedly made use of such diagrams to explain different solutions along with their respective unique implications.

We then turned our focus to the other significantly important types of Black Holes possible in nature starting with Charged Black holes without angular momentum described by the Reissner Nordstrom solution. This was followed by our discussion on purely rotating Black Holes and Charged Black Holes which possess angular momentum, the likes of which have been developed with the help of the Kerr solution. The idea of wormholes and possible space-time tunnels linking parallel universes or asymptotically flat regions of space-time have been elaborately explained as a consequence of Charged and Rotating Black Holes. We have carefully discussed each scenario of every individual solution in detail, along with their bizarre and amazing physical implications with the help of Penrose Diagrams. In the end we have shown how energy can be extracted from Rotating Black Holes by the Penrose Process of Energy extraction, which also serves as the basis for a thermodynamic approach to studying Black Holes.

However we must keep in mind that despite having such strong theoretical implications Black Holes are practically invisible and very hard to locate in nature. It took years of research and data extrapolation to figure out possible candidate Black Holes. The various methods adopted for its search vary from Gravitational Lensing to checking for intense gravitational influences on nearby stars (such as Sagittarius A in our Milky Way) to looking for the evidence of Accretion Discs surrounding Black Holes.

Today, astronomers have found convincing evidence for a supermassive black hole in the center of our own Milky Way galaxy, the galaxy NGC 4258, the giant elliptical galaxy M87, and several others. Scientists verified the existence of the black holes by studying the speed of the clouds of gas orbiting those regions. In 1994, Hubble Space Telescope data measured the mass of an unseen object at the center of M87. Based on the motion of the material whirling about the center, the object is estimated to be about 3 billion times the mass of our Sun and appears to be concentrated into a space smaller than our solar system.

For many years, X-ray emissions from the double-star system Cygnus X-1 convinced many astronomers that the system contains a black hole. With more precise measurements available recently, the evidence for a black hole in Cygnus X-1, and about a dozen other systems, is very strong and we can all look forward to a promising future with more experimental evidence of this bizarre enigma!

APPENDIX

Derivation of Einstein Field Equations from Einstein-Hilbert Action Principle:

All fundamental physical equations of the classical field including Einstein's field equations can be derived from The Variational Principle. The condition required in order to get the field equation follows from:

$$\boldsymbol{\delta \int \mathcal{L} d^4x = 0}$$

Of course the quantity above must be an invariant and must be constructed from the metric $\boldsymbol{g_{ab}}$ which is a dynamical variable in GR. We shall not include function which is first the derivative of metric because it vanishes at a point $P \in M$. The Riemann tensor is of course made from second derivative set of the metrics, and the only independent scalar constructed from the metric is the Ricci scalar R. The definition of Lagrangian density used here is $\boldsymbol{\mathcal{L} = \sqrt{-g}R}$, therefore:

$$\boldsymbol{S_{EH} = \int \sqrt{-g}R\, d^4x}$$

Where S_{EH} is the **Einstein-Hilbert Action**. We derive the field equations by variation of this action in the previous equation. So:

$$\boldsymbol{\delta S_{EH} = \delta \int \sqrt{-g}R d^4x}$$

$$\boldsymbol{= \int d^4x\, \delta^{3\sqrt{-g}g^{ab}} R_{ab}}$$

$$\boldsymbol{= \int d^4x\sqrt{-g}\, g^{ab}\delta R_{ab} + \int d^4x\sqrt{-g}\, R_{ab}\delta g^{ab} + \int d^4x R\delta\sqrt{-g}}$$

Now we have three terms of variation that are:

$$\boldsymbol{\delta S_{EH} = \delta S_{EH(1)} + \delta S_{EH(2)} + \delta S_{EH(3)}}$$

Let us now consider the variation of the first term:

$$\boldsymbol{\delta S_{EH(1)} = \int d^4x\sqrt{-g}\, R_{ab}\delta g^{ab}}$$

Considering the variation of the Ricci Tensor:

$$\boldsymbol{R_{ab} = R^c_{acb} = \partial_c\Gamma^c_{ab} - \partial_b\Gamma^c_{ac} + \Gamma^c_{cd}\Gamma^d_{ba} - \Gamma^c_{bd}\Gamma^d_{ac}}$$

$$\boldsymbol{\delta R_{ab} = \partial_c\delta\Gamma^c_{ab} - \partial_b\delta\Gamma^c_{ac} + \Gamma^d_{ba}\delta\Gamma^c_{cd} + \Gamma^c_{cd}\delta\Gamma^d_{ba} - \Gamma^d_{ac}\delta\Gamma^c_{bd} - \Gamma^c_{bd}\delta\Gamma^d_{ac}}$$

$$\boldsymbol{= (\partial_c\delta\Gamma^c_{ab} + \Gamma^c_{cd}\delta\Gamma^d_{ba} - \Gamma^d_{ac}\delta\Gamma^c_{bd} - \Gamma^d_{bc}\delta\Gamma^c_{ad}) - (\partial_b\delta\Gamma^c_{ac} + \Gamma^c_{bd}\delta\Gamma^d_{ac} - \Gamma^d_{ba}\delta\Gamma^c_{cd} - \Gamma^d_{bc}\delta\Gamma^c_{ad})}$$

and the covariant derivative formula is:

$$\nabla_c \delta\Gamma^c_{ab} = \partial_c \delta\Gamma^c_{ab} + \Gamma^c_{cd}\delta\Gamma^d_{ba} - \Gamma^d_{ac}\delta\Gamma^c_{bd} - \Gamma^d_{bc}\delta\Gamma^c_{ad}$$

and also:

$$\nabla_b \delta\Gamma^c_{ac} = \partial_b \delta\Gamma^c_{ac} + \Gamma^c_{bd}\delta\Gamma^d_{ac} - \Gamma^d_{ba}\delta\Gamma^c_{cd} - \Gamma^d_{bc}\delta\Gamma^c_{ad}$$

So we can conclude that:

$$\delta R_{ab} = \nabla_c \delta\Gamma^c_{ab} - \nabla_b \delta\Gamma^c_{ac}$$

Therefore the first part of the Einstein Hilbert action becomes:

$$\delta S_{EH(1)} = \int d^4x \sqrt{-g g^{ab}}\,(\nabla_c \delta\Gamma^c_{ab} - \nabla_b \delta\Gamma^c_{ac})$$

$$= \int d^4x \sqrt{-g}\,[\nabla_c (g^{ab}\delta\Gamma^c_{ab}) - \delta\Gamma^c_{ab}\nabla_c g^{ab} - \nabla_b (g^{ab}\nabla_b \delta\Gamma^c_{ac}) + \delta\Gamma^c_{ac}\nabla_b g^{ab}]$$

Remembering that the covariant derivative of the metric is zero, we get:

$$\delta S_{EH(1)} = \int d^4x \sqrt{-g g^{ab}}\,(\nabla_c \delta\Gamma^c_{ab} - \nabla_b \delta\Gamma^c_{ac})$$

$$= \int d^4x \sqrt{-g}\,[\nabla_c (g^{ab}\delta\Gamma^c_{ab}) - \nabla_b (g^{ab}\delta\Gamma^c_{ac})$$

$$= \int d^4x \sqrt{-g}\nabla_c\,[g^{ab}\delta\Gamma^c_{ab} - g^{ac}\delta\Gamma^b_{ab}]$$

This equation is an integral with respect to the volume element of the covariant divergence of a vector. Using Stokes's theorem we can easily see that this is equal to a boundary contribution at infinity which can be set to zero by vanishing the variation at infinity. Therefore this term contributes nothing to the total variation.

Now we shall carefully consider the variation of the metric g^{ab}. Since the contravariant and covariant metrics are symmetric matrices so,

$$g_{ca}g^{ab} = \delta^b_c$$

We now consider inverse of the metric: $g_{ab} = \frac{1}{g}\left(A^{ab}\right)^T = \frac{1}{g} A^{ab}$

Where, g is determinant and A^{ab} is the cofactor of the metric gab. Let us fix a, and expand the determinant g by the a^{th} row. Then:

$$g = g_{ab}A^{ab}$$

If we perform partial differentiation on both sides with respect to g_{ab}, then

$$\frac{\partial g}{\partial g_{ab}} = A^{ab}$$

Let us now consider the variation of determinant g such that:

$$\delta g = \frac{\partial g}{\partial g_{ab}} \delta g_{ab}$$

$$= A^{ab} \delta g_{ab}$$

$$= g g^{ba} \delta g_{ab}$$

Remembering thatg_{ab} is symmetric, we get: $\delta g = g g^{ba} \delta g_{ab}$

Using relation obtained above, we get: $\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\, \delta g$

$$= \frac{1}{2} \frac{g}{\sqrt{-g}} g^{ab} \delta g_{ab}$$

We shall convert from δg_{ab} to δg^{ab} by considering

$$\delta \delta_a^d = \delta(g_{ac} g^{cd}) = 0$$

$$g^{cd} \delta g_{ac} + g_{ac}\, \delta g^{cd} = 0$$

$$g^{cd} \delta g_{ac} = -\, g_{ac}\, \delta g^{cd}$$

Multiply both side of this equation by g_{bd} , we therefore have:

$$g_{bd} g^{cd} \delta g_{ac} = -g_{bd} g_{ac} \delta g^{cd}$$

$$\delta_b^c \delta g_{ac} = -g_{bd} g_{ac} \delta g^{cd}$$

$$\delta g_{ab} = -g_{ac} g_{bd} \delta g^{dc}$$

Substituting this in the previous relation we obtain:

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g^{ab} g_{ac} g_{bd} \delta g^{dc}$$

$$= -\frac{1}{2}\sqrt{-g}\, \delta_c^b g_{bd} \delta g^{dc}$$

$$= -\frac{1}{2}\sqrt{-g}\, g_{cd} \delta g^{dc}$$

Renaming indices c to a and d to b, we get

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g} g_{ab} \delta g^{ab}$$

So the variation of the Einstein Hilbert action becomes:

$$\delta S_{EH} = \int d^4x \sqrt{-g} R_{ab} \delta g^{ab} \; - \frac{1}{2} \int d^4x R \sqrt{-g} g_{ab} \delta g^{ab}$$

$$= \int d^4x \sqrt{-g} \;\; [\, R_{ab} \; - \frac{1}{2} g_{ab} R]\, \delta g^{ab}$$

The functional derivative of the action satisfies

$$\delta S = \int \sum_i \left(\frac{\delta S}{\delta \phi^{\,i}} \delta \phi \right) d^4x$$

Where { $\delta\phi^{\,i}$} is a complete set of field varied. Now stationary points are those for which we see that $\frac{\delta S}{\delta \phi^{\,i}} = \mathbf{0}.$ We now obtain Einstein's equation in vacuum which includes only the gravitational part of the action but not matter-field part due to the presence of energy-momentum. The Vacuum Field equations are represented as follows:

$$\frac{1}{\sqrt{-g}} \frac{\delta S_{EH}}{\delta gab} = R_{ab} - \frac{1}{2} g_{ab} R = 0$$

To obtain the complete field equations, we assume that there are other fields present besides the gravitational field. The action is then represented as:

$$S = \frac{1}{16\pi G} S_{EH} + S_M$$

Where, S_M is the action due to the presence of matter. We shall now normalize the gravitational action such that we get the right answer. Following the above equation we have:

$$\sqrt{-g}\, \frac{\delta S}{\delta g^{ab}} = \frac{1}{16\pi G} \left(R_{ab} - \frac{1}{2} g_{ab} R \right) + \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta gab} = 0$$

We may now define the energy-momentum tensor as:

$$T_{ab} = -2 \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g_{ab}}$$

This allows us to arrive at the complete Einstein Field Equations:

$$R_{ab} - \frac{1}{2} R g_{ab} = 8\pi G T_{ab}$$

We can easily replace the indices **a** and **b** with $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ to rewrite the equations in the more conventional form as:

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}}$$

Derivation of The Geodesic Equation:

In chapter (3) we have described the Geodesic Equation and its applications. Here we deal with the derivation of this Geodesic Equation in detail.

We begin with the line element (as done in chapter(3)) such that: $ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$ —— ①

Now, we know that proper time is defined by $d\tau^2 = -ds^2$
(Here we assume that $c = 1$)

$$\therefore \tau_{AB} = \int_A^B \sqrt{-ds^2} = \int_A^B \sqrt{-g_{\alpha\beta}\, dx^\alpha dx^\beta}\,.$$

In order, to write this as an integral that we can compute, we consider a parameterized worldline: $x^\alpha = x^\alpha(\sigma)$, where the parameter $\sigma = 0$ at point A and $\sigma = 1$ at point B. Then we write:

$$\tau_{AB} = \int_0^1 \left[-g_{\alpha\beta} \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma}\right]^{1/2} \Big\{ d\sigma \equiv \int_0^1 L\left[\frac{dx^\alpha}{d\sigma}, x^\alpha\right] d\sigma \quad —(2)$$

Here we have introduced the Lagrangian, $L\left[\frac{dx^\alpha}{d\sigma}, x^\alpha\right]$
We also note that: $L = \frac{d\tau}{d\sigma}$

$\therefore$ For functions $f = f(\tau(\sigma))$, we have

$$\frac{df}{d\sigma} = \frac{df}{d\tau}\frac{d\tau}{d\sigma} = L\frac{df}{d\tau}$$

We will use this above relation to change derivatives with respect to our arbitrary parameter σ to derivatives with respect to proper time, τ.

Using Variational Principle, we obtain the Euler Lagrange equations in the form:

$$-\frac{d}{d\sigma}\left(\frac{dL}{\partial(\partial x^\gamma/d\sigma)}\right) + \frac{\partial L}{\partial x^\gamma} = 0 \quad — (3)$$

Now, rearranging the terms on the right-hand side and changing the dummy index α to δ we have:

$$g_{\alpha\gamma}\frac{d^2x^\alpha}{d\tau^2} = \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\left(\frac{dx^\alpha}{d\tau}\right)\left(\frac{dx^\beta}{d\tau}\right) - \frac{1}{2}\left(\frac{\partial g_{\alpha\gamma}}{\partial x^\beta} + \frac{\partial g_{\gamma\beta}}{\partial x^\alpha}\right)\frac{dx^\alpha}{d\tau}\left(\frac{dx^\beta}{d\tau}\right)$$

$$= -\frac{1}{2}\left[\frac{\partial g_{\alpha\gamma}}{\partial x^\beta} + \frac{\partial g_{\gamma\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\right]\left(\frac{dx^\alpha}{d\tau}\right)\left(\frac{dx^\beta}{d\tau}\right)$$

$$= -\frac{1}{2}\left[\frac{\partial g_{\delta\gamma}}{\partial x^\beta} + \frac{\partial g_{\gamma\beta}}{\partial x^\delta} - \frac{\partial g_{\delta\beta}}{\partial x^\gamma}\right]\frac{dx^\delta}{d\tau}\frac{dx^\beta}{d\tau}$$

$$= -g_{\alpha\gamma}\Gamma^\alpha_{\delta\beta}\left[\frac{dx^\delta}{d\tau}\frac{dx^\beta}{d\tau}\right] \quad - (8)$$

$\therefore$ We finally arrive at the geodesic equation; which is of the form:

$$\frac{d^2x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\delta\beta}\left[\frac{dx^\delta}{d\tau}\frac{dx^\beta}{d\tau}\right] \quad - (9)$$

where the christoffel symbols satisfy the relation:

$$g_{\alpha\gamma}\Gamma^\alpha_{\delta\beta} = \frac{1}{2}\left[\frac{\partial g_{\gamma\delta}}{\partial x^\beta} + \frac{\partial g_{\gamma\beta}}{\partial x^\delta} - \frac{\partial g_{\delta\beta}}{\partial x^\gamma}\right] \quad - (10)$$

The SCHWARZSCHILD METRIC DERIVATION:

From Chapter (4)

We define an alternative Lagrangian in the following way:

$$L = \frac{1}{2} g_{\alpha\beta} \dot{x}^{\alpha} \dot{x}^{\beta} \quad - (1)$$

for which the metric

$$g_{\alpha\beta} = \begin{pmatrix} -e^{N(r)} & e^{P(r)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \quad - (2)$$

becomes ($x^0 \to t$, $x^1 \to r$, $x^2 \to \theta$, $x^3 \to \phi$):

$$L = -\frac{1}{2} e^{N} \dot{t}^2 + \frac{1}{2} e^{P} \dot{r}^2 + \frac{1}{2} r^2 \dot{\theta}^2 + \frac{1}{2} r^2 \sin^2\theta \, \dot{\phi}^2$$

This alternative Lagrangian allows us to compare with the Christoffel symbols of the Geodesic equation (derived before) which we can combine to obtain the Riemann & the Ricci Tensors. Now we shall solve the Lagrange Equations:

$$\frac{\partial L}{\partial x^{\alpha}} - \frac{d}{ds}\left(\frac{\partial L}{\partial \dot{x}^{\alpha}}\right) = 0 \quad - (3)$$

Now taking derivatives w.r.t. 't' components in space-time we get:

$$\frac{\partial L}{\partial t} - \frac{d}{ds}\left(\frac{\partial L}{\partial \dot{t}}\right) = 0$$

$$\Rightarrow 0 - \frac{d}{ds}\left(-e^{N}\dot{t}\right) = 0$$

$$\Rightarrow e^{N} \frac{dN}{dr} \frac{dr}{ds} \dot{t} + e^{N} \ddot{t} = 0$$

$$\Rightarrow e^{N} \left(\ddot{t} + N' \dot{t} \dot{r}\right) = 0$$

$$\Rightarrow \frac{d^2 t}{ds^2} + N' \left(\frac{dt}{ds}\right)\left(\frac{dr}{ds}\right) = 0 \quad - (4)$$

After comparing this to the geodesic equation, we get:

$$\frac{d^2 t}{ds^2} + \left(\Gamma^{0}_{01} + \Gamma^{0}_{10}\right)\left(\frac{dt}{ds}\right)\left(\frac{dr}{ds}\right) = 0 \quad - (5)$$

which means, due to the symmetry of Christoffel Symbols, $\Gamma^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\gamma\beta}$ we get: $\Gamma^{0}_{01} = \Gamma^{0}_{10} = \frac{1}{2} N'$

while other $\Gamma^{0}_{\alpha\beta}$ symbols vanish.

Now; w.r.t. the r- component we get: $\left(\frac{\partial L}{\partial r}\right) - \frac{d}{ds}\left(\frac{\partial L}{\partial \dot{r}}\right) = 0$

$$\Rightarrow -\tfrac{1}{2}N'e^{N}\dot{t}^2 + \tfrac{1}{2}\rho'e^{\rho}\dot{r}^2 + r\dot{\theta}^2 + r\sin^2\theta\,\dot{\phi}^2 - \frac{d}{ds}\left(e^{\rho}\dot{r}\right) = 0$$

$$\Rightarrow -\tfrac{1}{2}N'e^{N}\dot{t}^2 + \tfrac{1}{2}\rho'e^{\rho}\dot{r}^2 + r\dot{\theta}^2 + r\sin^2\theta\,\dot{\phi}^2 - e^{\rho}\rho'\dot{r}^2 - e^{\rho}\ddot{r} = 0$$

$$\Rightarrow -e^{\rho}\left(\ddot{r} + \tfrac{1}{2}N'e^{N-\rho}\dot{t}^2 + \tfrac{1}{2}\rho'\dot{r}^2 - e^{-\rho}r\dot{\theta}^2 - e^{-\rho}r\sin^2\theta\,\dot{\phi}^2\right) = 0$$

$$\Rightarrow \frac{d^2r}{ds^2} + \frac{1}{2}N'e^{N-\rho}\left(\frac{dt}{ds}\right)^2 + \frac{1}{2}\rho'\left(\frac{dr}{ds}\right)^2 - e^{-\rho}r\left(\frac{d\theta}{ds}\right)^2 - e^{-\rho}r\sin^2\theta\left(\frac{d\phi}{ds}\right)^2 = 0 \quad - (6)$$

After comparing to the Geodesic equation we obtain:

$$\frac{d^2r}{ds^2} + \Gamma^{1}_{00}\left(\frac{dt}{ds}\right)^2 + \Gamma^{1}_{11}\left(\frac{dr}{ds}\right)^2 + \Gamma^{1}_{22}\left(\frac{d\theta}{ds}\right)^2 + \Gamma^{1}_{33}\left(\frac{d\phi}{ds}\right)^2 = 0 \quad - (7)$$

which means:

$$\Gamma^{1}_{00} = \tfrac{1}{2}N'e^{N-\rho}$$

$$\Gamma^{1}_{11} = \tfrac{1}{2}\rho'$$

$$\Gamma^{1}_{22} = -e^{-\rho}r$$

$$\Gamma^{1}_{33} = -e^{-\rho}r\sin^2\theta$$

while other $\Gamma^{1}_{\alpha\beta}$ symbols vanish.

Now; w.r.t the θ component we get:

$$\frac{\partial L}{\partial \theta} = -\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0$$

$$\Rightarrow \frac{1}{2}r^2\, 2\sin\theta\cos\theta\,\dot{\phi}^2 - \frac{d}{ds}\left(r^2\dot{\theta}\right) = 0$$

$$\Rightarrow \tfrac{1}{2}r^2\sin 2\theta\,\dot{\phi}^2 - 2r\dot{r}\dot{\theta} - r^2\ddot{\theta} = 0$$

$$-r^2\left(\ddot{\theta} - \tfrac{1}{2}\sin 2\theta\,\dot{\phi}^2 - 2r\dot{r}\dot{\theta} - r^2\ddot{\theta}\right) = 0$$

$$-r^2\left(\ddot{\theta} - \tfrac{1}{2}\sin 2\theta\,\dot{\phi}^2 + 2\frac{\dot{r}}{r}\dot{\theta}\right) = 0$$

$$\frac{d^2\theta}{ds^2} + \frac{2}{r}\left(\frac{dr}{ds}\right)\left(\frac{d\theta}{ds}\right) - \tfrac{1}{2}\sin 2\theta\left(\frac{d\phi}{ds}\right)^2 = 0 \quad - (8)$$

After comparing this result to the geodesic equation we get:

$$\frac{d^2\theta}{ds^2} + \left(\Gamma^2_{12} + \Gamma^2_{21}\right)\left(\frac{dr}{ds}\right)\left(\frac{d\theta}{ds}\right) + \Gamma^2_{33}\left(\frac{d\phi}{ds}\right)^2 = 0 \quad -(9)$$

which means that:

$$\Gamma^2_{12} = \Gamma^2_{21} = 1/r$$

And $\Gamma^2_{33} = -1/2 \sin 2\theta$ — (10) while other $\Gamma^2_{\alpha\beta}$ symbols vanish.

Now; w.r.t. the ϕ component we get: $\frac{\partial L}{\partial \phi} - \frac{d}{ds}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = 0$

$$\Rightarrow 0 - \frac{d}{ds}\left(r^2 \sin^2\theta\, \dot{\phi}\right) = 0$$

$$\Rightarrow 2r\dot{r}\sin^2\theta\,\dot{\phi} - 2r^2\sin\theta\cos\theta\,\dot{\theta}\dot{\phi} - r^2\sin^2\theta\,\ddot{\phi} = 0$$

$$\Rightarrow -r^2\sin^2\theta\left(\ddot{\phi} + 2\frac{\dot{r}}{r}\dot{\phi} + 2\left(\frac{\cos\theta}{\sin\theta}\right)\dot{\theta}\dot{\phi}\right) = 0$$

$$\Rightarrow \frac{d^2\phi}{ds^2} + \frac{2}{r}\left(\frac{dr}{ds}\right)\left(\frac{d\phi}{ds}\right) + 2\cot\theta\left(\frac{d\theta}{ds}\right)\left(\frac{d\phi}{ds}\right) = 0 \quad -(11)$$

After comparing this to the Geodesic equation we obtain:

$$\frac{d^2\phi}{ds^2} + \left(\Gamma^3_{13} + \Gamma^3_{31}\right)\left(\frac{dr}{ds}\right)\left(\frac{d\phi}{ds}\right) + \left(\Gamma^3_{23} + \Gamma^3_{32}\right)\left(\frac{d\theta}{ds}\right)\left(\frac{d\phi}{ds}\right) = 0 \quad -(12)$$

this means that: $\Gamma^3_{13} = \Gamma^3_{31} = 1/r$

And $\Gamma^3_{23} = \Gamma^3_{32} = \cot\theta$ — (13)

while other $\Gamma^3_{\alpha\beta}$ symbols vanish. These Christoffel symbols associated with the metric $g_{\alpha\beta}$ are needed to compute the Reimann tensor which is in turn used to calculate the Ricci Tensor and the Ricci Scalar, to fully determine the LHS of Einstein's equation: $G_{\alpha\beta} \equiv R_{\alpha\beta} - 1/2\, g_{\alpha\beta} R = 0$

from $G_{\alpha\beta} = 0$ we get:

$$-\frac{e^{N-P}}{r}\left(P' - 1/r\right) - \frac{e^N}{r^2} = 0$$

$$\Rightarrow -N'/r - \frac{1}{r^2}\left(1 - e^P\right) = 0$$

$$\Rightarrow \frac{1}{2} r^2 e^{-P}\left[N'' - \frac{1}{2}P'N + \frac{1}{2}(N')^2 + \left(\frac{N' - P'}{r}\right)\right] = 0 \quad \text{— (14)}$$

Now these expressions combine to give :

$$\frac{dP}{dr} = -\frac{dN}{dr} = \frac{1}{r}\left(1 - e^{P}\right)$$

which can be solved for P in the following way:

$$\int \frac{dP}{1-e^P} = \int \frac{dr}{r} \Rightarrow \int\left(\frac{1-e^P}{1-e^P} + \frac{e^P}{1-e^P}\right)dP = \ln cr$$

where c is constant

$$\Rightarrow P - \ln(1-e^P) = \ln e^P - \ln(1-e^P) \Rightarrow \ln\left(\frac{e^P}{1-e^P}\right) = \ln cr$$

So we get : $\left(\frac{e^P}{1-e^P}\right) = cr \Rightarrow e^P = \left(\frac{cr}{1+cr}\right)$ — (15)

Solving for N we obtain : $N = -P + \text{const.} \Rightarrow e^N = e^{\text{const}} e^{-P}$

But ∵ we consider the Minkowski metric at large distances, we get : $\lim_{r\to\infty} g_{00} \to -1$ and $\lim_{r\to\infty} g_{11} \to 1$

And the const = 0 ∴ $N = -P$ so; $g_{00} = -e^N = -e^{-P}$

$$\Rightarrow g_{00} = -\left(\frac{1}{g_{11}}\right) = -\left(\frac{1+cr}{cr}\right) = -\left(1 + \frac{1}{cr}\right) \quad \text{— (16)}$$

For weak gravitational fields form its Newtonian Potential ϕ we get:

$$g_{00} = -\left(1 + \frac{2\phi}{c^2}\right) = -\left(1 - \frac{2GM}{rc^2}\right) = -\frac{1}{g_{11}}$$

Thus; we get $c = -\frac{c^2}{2GM}$ — (17)

This is how we finally arrive at the solution to the Schwarzschild problem with line element

$$ds^2 = -e^{N(r)}dt^2 + e^{P(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

as stated in chapter (4) and the final resultant metric we get is : <u>Schwarzschild Metric</u> :

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2dt^2 + \frac{dr^2}{\left(1 - \frac{2GM}{rc^2}\right)} + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2$$

this result has been used in chapter (4).

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